# YEARLY PAST PAPERS
## WITH TOPICAL QUESTIONS TRACKER

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**Cambridge AS & A Level**

# Mathematics (9709) Paper 3

**[Pure Maths 3]**

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# SAMPLE EDITION
## 2021 QUESTION PAPERS & MARK SCHEMES

*Note:*

**Provided for preview purposes to demonstrate format, organisation, and content quality. The full edition contains all examination years listed. For more information, visit:**
**Cambridge International AS & A Level Mathematics (9709) Product Page**

# Introduction

## Topical Questions Tracker: An Efficient Study Tool

A Topical Questions Tracker is a powerful tool, integrated as an appendix to Yearly Past Papers, enhancing the functionality of utilising Past Papers. This tool allows you to locate specific questions by topic, providing a much more efficient way to study for exams.

## How the Topical Questions Tracker Works

The Topical Questions Tracker is organized according to the latest syllabus of the subject. Each topic includes a comprehensive list of all relevant questions from the yearly past papers. For each question, the tracker provides:

- The paper's code
- The question number
- The sub-question number (if applicable)
- The page number where the question is located, which is hyperlinked for easy navigation

By clicking on the linked page number, you can jump directly to the corresponding page in the document, making it quick and simple to find the exact question you're looking for.

## Advantages of Topical Questions Trackers Over Traditional Past Papers

While traditional Topical Past Papers classify entire questions under a single topic, many questions contain sub-questions that may cover different topics. This can make it difficult to find specific practice material for a particular area of study.

The Topical Questions Tracker overcomes this limitation by categorizing each sub-question individually. This precise classification ensures that each part of the question is assigned to the appropriate topic, providing a more targeted and effective revision tool.

## Efficient Navigation Tips

While the Topical Questions Tracker allows you to jump directly to specific questions by clicking on the linked page numbers, navigating back to the previous page to find the next question can be time-consuming and somewhat frustrating. To streamline this process, you can utilize the 'Previous View' and 'Next View' commands in Adobe Reader.

To access these commands, navigate to the menu and select View » Go to » Previous View or Next View. Alternatively, you can use the shortcut keys for quicker navigation:

- Previous View: ALT + Left Arrow
- Next View: ALT + Right Arrow

These commands enable you to seamlessly move back and forth between the last two pages visited, enhancing your study efficiency by minimizing unnecessary navigation steps.

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Cambridge Assessment International Education

# Cambridge International AS & A Level

CANDIDATE NAME

CENTRE NUMBER

CANDIDATE NUMBER

*8483881097*

**MATHEMATICS** **9709/32**

Paper 3 Pure Mathematics 3 **February/March 2021**

**1 hour 50 minutes**

You must answer on the question paper.

You will need: List of formulae (MF19)

**INSTRUCTIONS**

- Answer **all** questions.
- Use a black or dark blue pen. You may use an HB pencil for any diagrams or graphs.
- Write your name, centre number and candidate number in the boxes at the top of the page.
- Write your answer to each question in the space provided.
- Do **not** use an erasable pen or correction fluid.
- Do **not** write on any bar codes.
- If additional space is needed, you should use the lined page at the end of this booklet; the question number or numbers must be clearly shown.
- You should use a calculator where appropriate.
- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

**INFORMATION**

- The total mark for this paper is 75.
- The number of marks for each question or part question is shown in brackets [ ].

This document has **20** pages. Any blank pages are indicated.

**BLANK PAGE**

**1** Solve the equation $\ln(x^3 - 3) = 3\ln x - \ln 3$. Give your answer correct to 3 significant figures. [3]

**2** The polynomial $ax^3 + 5x^2 - 4x + b$, where $a$ and $b$ are constants, is denoted by $\mathrm{p}(x)$. It is given that $(x + 2)$ is a factor of $\mathrm{p}(x)$ and that when $\mathrm{p}(x)$ is divided by $(x + 1)$ the remainder is 2.

Find the values of $a$ and $b$. [5]

**3** By first expressing the equation $\tan(x + 45°) = 2\cot x + 1$ as a quadratic equation in $\tan x$, solve the equation for $0° < x < 180°$. [6]

**4** The variables $x$ and $y$ satisfy the differential equation

$$(1 - \cos x)\frac{\mathrm{d}y}{\mathrm{d}x} = y \sin x.$$

It is given that $y = 4$ when $x = \pi$.

**(a)** Solve the differential equation, obtaining an expression for $y$ in terms of $x$. [6]

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**(b)** Sketch the graph of $y$ against $x$ for $0 < x < 2\pi$. [1]

**5** **(a)** Express $\sqrt{7}\sin x + 2\cos x$ in the form $R\sin(x + \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. State the exact value of $R$ and give $\alpha$ correct to 2 decimal places. [3]

**(b)** Hence solve the equation $\sqrt{7}\sin 2\theta + 2\cos 2\theta = 1$, for $0° < \theta < 180°$. [5]

**6** Let $f(x) = \dfrac{5a}{(2x-a)(3a-x)}$, where $a$ is a positive constant.

**(a)** Express $f(x)$ in partial fractions. [3]

**(b)** Hence show that $\int_a^{2a} \mathrm{f}(x)\,\mathrm{d}x = \ln 6$. [4]

**7** Two lines have equations $\mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + s \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}$.

**(a)** Show that the lines are skew. [5]

**(b)** Find the acute angle between the directions of the two lines. [3]

**8** The complex numbers $u$ and $v$ are defined by $u = -4 + 2\mathrm{i}$ and $v = 3 + \mathrm{i}$.

**(a)** Find $\frac{u}{v}$ in the form $x + \mathrm{i}y$, where $x$ and $y$ are real. [3]

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**(b)** Hence express $\frac{u}{v}$ in the form $re^{\mathrm{i}\theta}$, where $r$ and $\theta$ are exact. [2]

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In an Argand diagram, with origin $O$, the points $A$, $B$ and $C$ represent the complex numbers $u$, $v$ and $2u + v$ respectively.

**(c)** State fully the geometrical relationship between $OA$ and $BC$. [2]

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**(d)** Prove that angle $AOB = \frac{3}{4}\pi$. [2]

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**9** Let $f(x) = \dfrac{e^{2x} + 1}{e^{2x} - 1}$, for $x > 0$.

**(a)** The equation $x = f(x)$ has one root, denoted by $a$.

Verify by calculation that $a$ lies between 1 and 1.5. [2]

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**(b)** Use an iterative formula based on the equation in part **(a)** to determine $a$ correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

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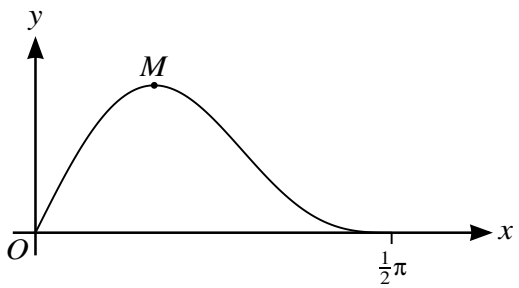
**(c)** Find $f'(x)$. Hence find the exact value of $x$ for which $f'(x) = -8$. [6]

**10**




The diagram shows the curve $y = \sin 2x \cos^2 x$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$, and its maximum point $M$.

**(a)** Using the substitution $u = \sin x$, find the exact area of the region bounded by the curve and the $x$-axis. [5]

**(b)** Find the exact $x$-coordinate of $M$. [6]

## Additional Page

If you use the following lined page to complete the answer(s) to any question(s), the question number(s) must be clearly shown.

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Cambridge Assessment International Education

# Cambridge International A Level

**MATHEMATICS** **9709/32**

Paper 3 Pure Mathematics 3 **March 2021**

MARK SCHEME

Maximum Mark: 75

**Published**

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

Cambridge International will not enter into discussions about these mark schemes.

Cambridge International is publishing the mark schemes for the March 2021 series for most Cambridge IGCSE™, Cambridge International A and AS Level components and some Cambridge O Level components.





**[Turn over**

**Generic Marking Principles**

These general marking principles must be applied by all examiners when marking candidate answers. They should be applied alongside the specific content of the mark scheme or generic level descriptors for a question. Each question paper and mark scheme will also comply with these marking principles.

GENERIC MARKING PRINCIPLE 1:
Marks must be awarded in line with:

- the specific content of the mark scheme or the generic level descriptors for the question
- the specific skills defined in the mark scheme or in the generic level descriptors for the question
- the standard of response required by a candidate as exemplified by the standardisation scripts.

GENERIC MARKING PRINCIPLE 2:
Marks awarded are always **whole marks** (not half marks, or other fractions).

GENERIC MARKING PRINCIPLE 3:
Marks must be awarded **positively**:

- marks are awarded for correct/valid answers, as defined in the mark scheme. However, credit is given for valid answers which go beyond the scope of the syllabus and mark scheme, referring to your Team Leader as appropriate
- marks are awarded when candidates clearly demonstrate what they know and can do
- marks are not deducted for errors
- marks are not deducted for omissions
- answers should only be judged on the quality of spelling, punctuation and grammar when these features are specifically assessed by the question as indicated by the mark scheme. The meaning, however, should be unambiguous.

GENERIC MARKING PRINCIPLE 4:
Rules must be applied consistently, e.g. in situations where candidates have not followed instructions or in the application of generic level descriptors.

GENERIC MARKING PRINCIPLE 5:
Marks should be awarded using the full range of marks defined in the mark scheme for the question (however; the use of the full mark range may be limited according to the quality of the candidate responses seen).

GENERIC MARKING PRINCIPLE 6:
Marks awarded are based solely on the requirements as defined in the mark scheme. Marks should not be awarded with grade thresholds or grade descriptors in mind.

| Mathematics Specific Marking Principles | |
|---|---|
| 1 | Unless a particular method has been specified in the question, full marks may be awarded for any correct method. However, if a calculation is required then no marks will be awarded for a scale drawing. |
| 2 | Unless specified in the question, answers may be given as fractions, decimals or in standard form. Ignore superfluous zeros, provided that the degree of accuracy is not affected. |
| 3 | Allow alternative conventions for notation if used consistently throughout the paper, e.g. commas being used as decimal points. |
| 4 | Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored (isw). |
| 5 | Where a candidate has misread a number in the question and used that value consistently throughout, provided that number does not alter the difficulty or the method required, award all marks earned and deduct just 1 mark for the misread. |
| 6 | Recovery within working is allowed, e.g. a notation error in the working where the following line of working makes the candidate's intent clear. |

**Mark Scheme Notes**

The following notes are intended to aid interpretation of mark schemes in general, but individual mark schemes may include marks awarded for specific reasons outside the scope of these notes.

**Types of mark**

**M** Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.

**A** Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).

**B** Mark for a correct result or statement independent of method marks.

**DM** or **DB** When a part of a question has two or more 'method' steps, the M marks are generally independent unless the scheme specifically says otherwise; and similarly, when there are several B marks allocated. The notation DM or DB is used to indicate that a particular M or B mark is dependent on an earlier M or B (asterisked) mark in the scheme. When two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.

**FT** Implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A or B marks are given for correct work only.

- A or B marks are given for correct work only (not for results obtained from incorrect working) unless follow through is allowed (see abbreviation FT above).
- For a numerical answer, allow the A or B mark if the answer is correct to 3 significant figures or would be correct to 3 significant figures if rounded (1 decimal place for angles in degrees).
- The total number of marks available for each question is shown at the bottom of the Marks column.
- Wrong or missing units in an answer should not result in loss of marks unless the guidance indicates otherwise.
- Square brackets [ ] around text or numbers show extra information not needed for the mark to be awarded.

**Abbreviations**

| | |
|---|---|
| AEF/OE | Any Equivalent Form (of answer is equally acceptable) / Or Equivalent |
| AG | Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid) |
| CAO | Correct Answer Only (emphasising that no 'follow through' from a previous error is allowed) |
| CWO | Correct Working Only |
| ISW | Ignore Subsequent Working |
| SOI | Seen Or Implied |
| SC | Special Case (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance) |
| WWW | Without Wrong Working |
| AWRT | Answer Which Rounds To |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 1 | Use law of the logarithm of a product or power | **M1** | |
| | Obtain a correct equation free of logarithms, e.g. $3\left(x^3-3\right)=x^3$ | **A1** | |
| | Obtain $x = 1.65$ | **A1** | |
| | | **3** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 2 | Substitute $x = -2$, equate result to zero and obtain a correct equation, e.g. $-8a+20+8+b=0$ | **B1** | |
| | Substitute $x = -1$ and equate result to 2 | **M1** | |
| | Obtain a correct equation, e.g. $-a+5+4+b=2$ | **A1** | |
| | Solve for $a$ or for $b$ | **M1** | |
| | Obtain $a = 3$ and $b = -4$ | **A1** | |
| | | **5** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 3 | Use correct trig formulae to obtain an equation in tan $x$ | ***M1** | |
| | Using $\tan 45° = 1$, obtain a horizontal equation in $\tan x$ in any form | **DM1** | |
| | Reduce the equation to $\tan^2 x + \tan x - 1 = 0$, or 3-term equivalent | **A1** | |
| | Solve a 3-term quadratic in tan $x$, for $x$ | **M1** | |
| | Obtain answer, e.g. $x = 31.7°$ | **A1** | |
| | Obtain second answer, e.g. $x = 121.7°$, and no other in the interval | **A1** | Ignore answers outside the given interval. |
| | | **6** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 4(a) | Separate variables correctly and attempt integration of at least one side | **M1** | |
| | Obtain term $\ln y$ | **A1** | |
| | Obtain term of the form $\pm\ln(1-\cos x)$ | **M1** | |
| | Obtain term $\ln(1-\cos x)$ | **A1** | |
| | Use $x = \pi$, $y = 4$ to evaluate a constant, or as limits, in a solution containing terms of the form $a\ln y$ and $b\ln(1-\cos x)$ | **M1** | |
| | Obtain final answer $y = 2(1-\cos x)$ | **A1** | OE |
| | | **6** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 4(b) | Show a correct graph for $0 < x < 2\pi$ with the maximum at $x = \pi$ | **B1 FT** | The FT is for graphs of the form $y = a(1 - \cos x)$, where $a$ is positive. |
| | | **1** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 5(a) | State $R = \sqrt{11}$ | **B1** | |
| | Use trig formulae to find $\alpha$ | **M1** | |
| | Obtain $\alpha = 37.09°$ | **A1** | |
| | | **3** | |
| 5(b) | Evaluate $\sin^{-1}\left(\frac{1}{\sqrt{11}}\right)$ to at least 2 dp (17.5484°) | **B1 FT** | The FT is on $R$. |
| | Use correct method to find a value of $\theta$ in the interval | **M1** | |
| | Obtain answer, e.g. 62.7° | **A1** | |
| | Use a correct method to obtain a second answer | **M1** | |
| | Obtain second answer, e.g. 170.2° , and no other in the interval | **A1** | Ignore answers outside the given interval. |
| | | **5** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 6(a) | Carry out a relevant method to determine constants $A$ and $B$ such that $\frac{5a}{(2x-a)(3a-x)} = \frac{A}{2x-a} + \frac{B}{3a-x}$ | **M1** | |
| | Obtain $A = 2$ | **A1** | |
| | Obtain $B = 1$ | **A1** | |
| | | **3** | |
| 6(b) | Integrate and obtain terms $\ln(2x-a) - \ln(3a-x)$ | **B1 FT**<br>**B1 FT** | The FT is on the values of $A$ and $B$. |
| | Substitute limits correctly in a solution containing terms of the form $b\ln(2x-a)$ and $c\ln(3a-x)$, where $bc \neq 0$ | **M1** | |
| | Obtain the given answer showing full and correct working | **A1** | |
| | | **4** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 7(a) | Express general point of a line in component form, e.g. $(1 + 2s, 3 - s, 2 + 3s)$ or $(2 + t, 1 - t, 4 + 4t)$ | **B1** | |
| | Equate at least two pairs of components and solve for $s$ or for $t$ | **M1** | |
| | Obtain correct answer for $s$ or for $t$ (possible answers are $-1$, 6, $\frac{2}{5}$ for $s$ and $-3$, 4, $-\frac{1}{5}$ for $t$ ) | **A1** | |
| | Verify that all three component equations are not satisfied | **A1** | |
| | Show that the lines are not parallel and are thus skew | **A1** | |
| | | **5** | |
| 7(b) | Carry out correct process for evaluating the scalar product of the direction vectors | **M1** | |
| | Using the correct process for the moduli, divide the scalar product by the product of the moduli and evaluate the inverse cosine of the result | **M1** | |
| | Obtain answer 19.1° or 0.333 radians | **A1** | |
| | | **3** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 8(a) | Multiply numerator and denominator by 3 – i | **M1** | OE |
| | Obtain numerator – 10 + 10i or denominator 10 | **A1** | |
| | Obtain final answer – 1 + i | **A1** | |
| | | **3** | |
| 8(b) | State or imply $r = \sqrt{2}$ | **B1 FT** | |
| | State or imply that $\theta = \frac{3}{4}\pi$ | **B1 FT** | |
| | | **2** | |
| 8(c) | State that $OA$ and $BC$ are parallel | **B1** | |
| | State that $BC = 2OA$ | **B1** | |
| | | **2** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 8(d) | Use angle $AOB = \arg u - \arg v = \arg\frac{u}{v}$ | **M1** | |
| | Obtain the given answer | **A1** | |
| | **Alternative method for question 8(d)** | | |
| | Obtain tan $AOB$ from gradients of $OA$ and $OB$ and the $\tan(A \pm B)$ formula | **M1** | |
| | Obtain the given answer | **A1** | |
| | **Alternative method for question 8(d)** | | |
| | Obtain $\cos AOB$ by using the cosine rule or a scalar product | **M1** | |
| | Obtain the given answer | **A1** | |
| | | **2** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 9(a) | Calculate the values of a relevant expression or pair of expressions at $x = 1$ and $x = 1.5$ | **M1** | |
| | Complete the argument correctly with correct calculated values | **A1** | |
| | | **2** | |
| 9(b) | Use the iterative formula $x_{n+1} = \frac{e^{2x_n}+1}{e^{2x_n}-1}$, or equivalent, correctly at least once | **M1** | |
| | Obtain final answer 1.20 | **A1** | |
| | Show sufficient iterations to 4 dp to justify 1.20 to 2 dp, or show there is a sign change in the interval (1.195,1.205) | **A1** | |
| | | **3** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 9(c) | Use quotient rule | **M1** | |
| | Obtain correct derivative in any form | **A1** | |
| | Equate derivative to – 8 and obtain a quadratic in $e^{2x}$ | **M1** | |
| | Obtain $2\left(e^{2x}\right)^2 - 5e^{2x} + 2 = 0$ | **A1** | OE |
| | Solve a 3-term quadratic in $e^{2x}$ for $x$ | **M1** | |
| | Obtain answer $x = \frac{1}{2}\ln 2$, or exact equivalent, only | **A1** | |
| | **Alternative method for question 9(c)** | | |
| | Use quotient rule | **M1** | |
| | Obtain correct derivative in any form | **A1** | |
| | Equate derivative to – 8, take square roots and obtain a quadratic in $e^{x}$ | **M1** | |
| | Obtain $\sqrt{2}e^{2x} - e^{x} - \sqrt{2} = 0$ | **A1** | OE |
| | Solve a 3-term quadratic in $e^{x}$ for $x$ | **M1** | |
| | Obtain answer $x = \frac{1}{2}\ln 2$, or exact equivalent, only | **A1** | |
| | | **6** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 10(a) | State or imply $du = \cos x\, dx$ | **B1** | |
| | Using double angle formula for sin2$x$ and Pythagoras, express integral in terms of $u$ and d$u$. | **M1** | |
| | Obtain integral $\int 2\left(u - u^3\right)du$ | **A1** | OE |
| | Use limits $u = 0$ and $u = 1$ in an integral of the form $au^2 + bu^4$, where $ab \neq 0$ | **M1** | $a + b$ or $a + b - 0$ $\left(a = 1 \text{ and } b = -\frac{1}{2}\right)$ |
| | Obtain answer $\frac{1}{2}$ | **A1** | |
| | | **5** | |
| 10(b) | Use product rule | **M1** | |
| | Obtain correct derivative in any form | **A1** | |
| | Equate derivative to zero and use a double angle formula | ***M1** | |
| | Obtain an equation in one trig variable | **DM1** | |
| | Obtain $4\sin^2 x = 1$, $4\cos^2 x = 3$ or $3\tan^2 x = 1$ | **A1** | |
| | Obtain answer $x = \frac{1}{6}\pi$ | **A1** | |
| | | **6** | |

# 2 2021 | May/Jun | Variant 1 | 9709_s21_qp_31

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Cambridge Assessment International Education

# Cambridge International AS & A Level

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**MATHEMATICS** **9709/31**

Paper 3 Pure Mathematics 3 **May/June 2021**

**1 hour 50 minutes**

You must answer on the question paper.

You will need: List of formulae (MF19)

**INSTRUCTIONS**

- Answer **all** questions.
- Use a black or dark blue pen. You may use an HB pencil for any diagrams or graphs.
- Write your name, centre number and candidate number in the boxes at the top of the page.
- Write your answer to each question in the space provided.
- Do **not** use an erasable pen or correction fluid.
- Do **not** write on any bar codes.
- If additional space is needed, you should use the lined page at the end of this booklet; the question number or numbers must be clearly shown.
- You should use a calculator where appropriate.
- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

**INFORMATION**

- The total mark for this paper is 75.
- The number of marks for each question or part question is shown in brackets [ ].

This document has **20** pages. Any blank pages are indicated.

**1** Solve the inequality $2|3x - 1| < |x + 1|$. [4]

**2** Find the real root of the equation $\frac{2e^x + e^{-x}}{2 + e^x} = 3$, giving your answer correct to 3 decimal places. Your working should show clearly that the equation has only one real root. [5]

**3** **(a)** Given that $\cos(x - 30°) = 2\sin(x + 30°)$, show that $\tan x = \dfrac{2 - \sqrt{3}}{1 - 2\sqrt{3}}$. [4]

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**(b)** Hence solve the equation

$$\cos(x - 30°) = 2\sin(x + 30°),$$

for $0° < x < 360°$. [2]

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**4** **(a)** Prove that $\dfrac{1-\cos 2\theta}{1+\cos 2\theta} \equiv \tan^2\theta$. [2]

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**(b)** Hence find the exact value of $\displaystyle\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{1-\cos 2\theta}{1+\cos 2\theta}\,\mathrm{d}\theta$. [4]

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**5** **(a)** Solve the equation $z^2 - 2piz - q = 0$, where $p$ and $q$ are real constants. [2]

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In an Argand diagram with origin $O$, the roots of this equation are represented by the distinct points $A$ and $B$.

**(b)** Given that $A$ and $B$ lie on the imaginary axis, find a relation between $p$ and $q$. [2]

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**(c)** Given instead that triangle $OAB$ is equilateral, express $q$ in terms of $p$. [3]

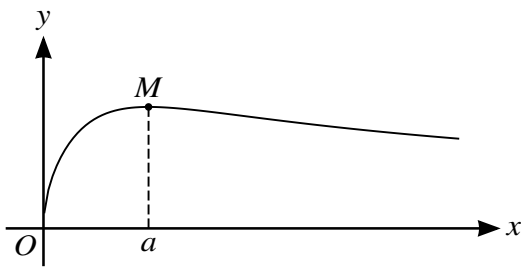
**6** The parametric equations of a curve are

$$x = \ln(2 + 3t), \qquad y = \frac{t}{2 + 3t}.$$

**(a)** Show that the gradient of the curve is always positive. [5]

**(b)** Find the equation of the tangent to the curve at the point where it intersects the $y$-axis. [3]

**7**




The diagram shows the curve $y = \dfrac{\tan^{-1} x}{\sqrt{x}}$ and its maximum point $M$ where $x = a$.

**(a)** Show that $a$ satisfies the equation

$$a = \tan\left(\frac{2a}{1 + a^2}\right).$$ [4]

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**(b)** Verify by calculation that $a$ lies between 1.3 and 1.5. [2]

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**(c)** Use an iterative formula based on the equation in part **(a)** to determine $a$ correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

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**8** With respect to the origin $O$, the points $A$ and $B$ have position vectors given by $\overrightarrow{OA} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$. The line $l$ has equation $\mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$.

**(a)** Find the acute angle between the directions of $AB$ and $l$. [4]

**(b)** Find the position vector of the point $P$ on $l$ such that $AP = BP$. [5]

**9** The equation of a curve is $y = x^{-\frac{2}{3}} \ln x$ for $x > 0$. The curve has one stationary point.

**(a)** Find the exact coordinates of the stationary point. [5]

**(b)** Show that $\int_1^8 y \, \mathrm{d}x = 18 \ln 2 - 9$. [5]

**10** The variables $x$ and $t$ satisfy the differential equation $\frac{\mathrm{d}x}{\mathrm{d}t} = x^2(1 + 2x)$, and $x = 1$ when $t = 0$.

Using partial fractions, solve the differential equation, obtaining an expression for $t$ in terms of $x$. [11]

## Additional Page

If you use the following lined page to complete the answer(s) to any question(s), the question number(s) must be clearly shown.

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Cambridge Assessment International Education

# Cambridge International A Level

**MATHEMATICS** **9709/31**

Paper 3 Pure Mathematics 3 **May/June 2021**

MARK SCHEME

Maximum Mark: 75

**Published**

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

Cambridge International will not enter into discussions about these mark schemes.

Cambridge International is publishing the mark schemes for the May/June 2021 series for most Cambridge IGCSE™, Cambridge International A and AS Level components and some Cambridge O Level components.





**[Turn over**

**PUBLISHED**

**Generic Marking Principles**

These general marking principles must be applied by all examiners when marking candidate answers. They should be applied alongside the specific content of the mark scheme or generic level descriptors for a question. Each question paper and mark scheme will also comply with these marking principles.

GENERIC MARKING PRINCIPLE 1:
Marks must be awarded in line with:

- the specific content of the mark scheme or the generic level descriptors for the question
- the specific skills defined in the mark scheme or in the generic level descriptors for the question
- the standard of response required by a candidate as exemplified by the standardisation scripts.

GENERIC MARKING PRINCIPLE 2:
Marks awarded are always **whole marks** (not half marks, or other fractions).

GENERIC MARKING PRINCIPLE 3:
Marks must be awarded **positively**:

- marks are awarded for correct/valid answers, as defined in the mark scheme. However, credit is given for valid answers which go beyond the scope of the syllabus and mark scheme, referring to your Team Leader as appropriate
- marks are awarded when candidates clearly demonstrate what they know and can do
- marks are not deducted for errors
- marks are not deducted for omissions
- answers should only be judged on the quality of spelling, punctuation and grammar when these features are specifically assessed by the question as indicated by the mark scheme. The meaning, however, should be unambiguous.

GENERIC MARKING PRINCIPLE 4:
Rules must be applied consistently, e.g. in situations where candidates have not followed instructions or in the application of generic level descriptors.

GENERIC MARKING PRINCIPLE 5:
Marks should be awarded using the full range of marks defined in the mark scheme for the question (however; the use of the full mark range may be limited according to the quality of the candidate responses seen).

GENERIC MARKING PRINCIPLE 6:
Marks awarded are based solely on the requirements as defined in the mark scheme. Marks should not be awarded with grade thresholds or grade descriptors in mind.

| Mathematics Specific Marking Principles | |
|---|---|
| 1 | Unless a particular method has been specified in the question, full marks may be awarded for any correct method. However, if a calculation is required then no marks will be awarded for a scale drawing. |
| 2 | Unless specified in the question, answers may be given as fractions, decimals or in standard form. Ignore superfluous zeros, provided that the degree of accuracy is not affected. |
| 3 | Allow alternative conventions for notation if used consistently throughout the paper, e.g. commas being used as decimal points. |
| 4 | Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored (isw). |
| 5 | Where a candidate has misread a number in the question and used that value consistently throughout, provided that number does not alter the difficulty or the method required, award all marks earned and deduct just 1 mark for the misread. |
| 6 | Recovery within working is allowed, e.g. a notation error in the working where the following line of working makes the candidate's intent clear. |

**Mark Scheme Notes**

The following notes are intended to aid interpretation of mark schemes in general, but individual mark schemes may include marks awarded for specific reasons outside the scope of these notes.

**Types of mark**

**M** Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.

**A** Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).

**B** Mark for a correct result or statement independent of method marks.

**DM** or **DB** When a part of a question has two or more 'method' steps, the M marks are generally independent unless the scheme specifically says otherwise; and similarly, when there are several B marks allocated. The notation DM or DB is used to indicate that a particular M or B mark is dependent on an earlier M or B (asterisked) mark in the scheme. When two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.

**FT** Implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A or B marks are given for correct work only.

- A or B marks are given for correct work only (not for results obtained from incorrect working) unless follow through is allowed (see abbreviation FT above).
- For a numerical answer, allow the A or B mark if the answer is correct to 3 significant figures or would be correct to 3 significant figures if rounded (1 decimal place for angles in degrees).
- The total number of marks available for each question is shown at the bottom of the Marks column.
- Wrong or missing units in an answer should not result in loss of marks unless the guidance indicates otherwise.
- Square brackets [ ] around text or numbers show extra information not needed for the mark to be awarded.

**Abbreviations**

| | |
|---|---|
| AEF/OE | Any Equivalent Form (of answer is equally acceptable) / Or Equivalent |
| AG | Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid) |
| CAO | Correct Answer Only (emphasising that no 'follow through' from a previous error is allowed) |
| CWO | Correct Working Only |
| ISW | Ignore Subsequent Working |
| SOI | Seen Or Implied |
| SC | Special Case (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance) |
| WWW | Without Wrong Working |
| AWRT | Answer Which Rounds To |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 1 | State or imply non-modular inequality $2^2(3x-1)^2 < (x+1)^2$, or corresponding quadratic equation, or pair of linear equations | **B1** | |
| | Form and solve a 3-term quadratic, or solve two linear equations for $x$ | **M1** | e.g. $35x^2 - 26x + 3 = 0$ |
| | Obtain critical values $x = \frac{3}{5}$ and $x = \frac{1}{7}$ | **A1** | Allow 0.143 or better |
| | State final answer $\frac{1}{7} < x < \frac{3}{5}$ | **A1** | Exact values required. Accept $x > \frac{1}{7}$ **and** $x < \frac{3}{5}$<br>Do not condone $\leqslant$ for $<$ in the final answer.<br>Fractions need not be in lowest terms. |
| | **Alternative method for Question 1** | | |
| | Obtain critical value $x = \frac{3}{5}$ from a graphical method, or by solving a linear equation or linear inequality | **B1** | |
| | Obtain critical value $x = \frac{1}{7}$ similarly | **B2** | Allow 0.143 or better |
| | State final answer $\frac{1}{7} < x < \frac{3}{5}$ | **B1** | OE. Exact values required. Accept $x > \frac{1}{7}$ **and** $x < \frac{3}{5}$<br>Do not condone $\leqslant$ for $<$ in the final answer.<br>Fractions need not be in lowest terms. |
| | | **4** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 2 | Reduce to a 3-term quadratic $u^2+6u-1=0$ OE | **B1** | Allow '= 0' implied |
| | Solve a 3-term quadratic for $u$ | **M1** | |
| | Obtain root $\sqrt{10}-3$ | **A1** | |
| | Obtain answer $x = -1.818$ only | **A1** | The question asks for 3 d.p. |
| | Reject $-\sqrt{10}-3$ correctly | **B1** | e.g. by stating that $e^x>0$ or $\ln\left(-10-\sqrt{3}\right)$ is impossible<br>Not "math error". |
| | **Alternative method for Question 2** | | |
| | Rearrange to obtain a correct iterative formula | **B1** | e.g. $x_{n+1}=-\ln\left(6+e^{x_n}\right)$ |
| | Use the iterative process at least twice | **M1** | |
| | Obtain answer $x = -1.818$ | **A1** | |
| | Show sufficient iterations to at least 4 d.p. to justify $x = -1.818$ | **A1** | 1, −2.165..., −1.811..., −1.819..., −1.818..., −1.818... |
| | Clear explanation of why there is only one real root | **B1** | |
| | | **5** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 3(a) | Use correct trig expansions and obtain an equation in $\sin x$ and $\cos x$ | ***M1** | |
| | Use correct exact trig ratios for $30°$ in *their* expansion | **B1 FT** | e.g. $\cos x\left(\frac{\sqrt{3}}{2}-1\right)=\sin x\left(\sqrt{3}-\frac{1}{2}\right)$ |
| | Obtain an equation in $\tan x$ | **DM1** | Allow if their error in line 1 was a sign error |
| | Obtain $\tan x=\frac{2-\sqrt{3}}{1-2\sqrt{3}}$ from correct working | **A1** | AG |
| | | **4** | |
| 3(b) | Obtain answer in the given interval, e.g.$173.8°$ | **B1** | Accept $174°$, $354°$ or better |
| | Obtain a second answer and no other in the given interval, e.g. $353.8°$ | **B1** | Ignore answers outside the given interval.<br>Treat answers in radians (3.03 and 6.17) as a misread. |
| | | **2** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 4(a) | Use correct double angle formula or *t*-substitution twice | **M1** | |
| | Obtain $\frac{1-\cos 2\theta}{1+\cos 2\theta}=\tan^2\theta$ from correct working | **A1** | AG |
| | | **2** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 4(b) | Express $\tan^2\theta$ in terms of $\sec^2\theta$ | **M1** | $\left(\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\left(\sec^2\theta \pm 1\right)\mathrm{d}\theta\right)$ |
| | Integrate and obtain terms $\tan\theta - \theta$ | **A1** | Accept with a mixture of $x$ and $\theta$ |
| | Substitute limits correctly in an integral of the form $a\tan\theta + b\theta$, where $ab \neq 0$ | **M1** | $\left(\sqrt{3} - \frac{\pi}{3} - \frac{1}{\sqrt{3}} + \frac{\pi}{6}\right)$ Allow if trig. not substituted |
| | Obtain answer $\frac{2}{3}\sqrt{3} - \frac{1}{6}\pi$ | **A1** | or equivalent exact 2-term expression |
| | | **4** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 5(a) | Use quadratic formula and $\mathrm{i}^2 = -1$ | **M1** | |
| | Obtain answers $p\mathrm{i} + \sqrt{q - p^2}$ and $p\mathrm{i} - \sqrt{q - p^2}$ | **A1** | Accept $\frac{2p\mathrm{i} \pm \sqrt{-4p^2 + 4q}}{2}$ and ISW |
| | | **2** | |
| 5(b) | State or imply that the discriminant must be negative | **M1** | |
| | State condition $q < p^2$ | **A1** | |
| | | **2** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 5(c) | Carry out a correct method for finding a relation, e.g. use the fact that the argument of one of the roots is $(\pm)60°$ | **M1** | |
| | State a correct relation in any form, e.g. $\frac{p}{\sqrt{q-p^2}} = (\pm)\sqrt{3}$ | **A1** | |
| | Simplify to $q = \frac{4}{3}p^2$ | **A1** | |
| | **Alternative method for Question 5(c)** | | |
| | Carry out a correct method for finding a relation, e.g. use the fact that the sides have equal length | **M1** | |
| | State a correct relation in any form, e.g. $4\left(q-p^2\right) = p^2 + q - p^2$ | **A1** | |
| | Simplify to $q = \frac{4}{3}p^2$ | **A1** | |
| | | **3** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 6(a) | Use correct chain rule **or** correct quotient rule to differentiate $x$ or $y$ | **M1** | |
| | Obtain $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{3}{2+3t}$ **or** $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{2}{(2+3t)^2}$ | **A1** | OE |
| | Use $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \div \frac{\mathrm{d}x}{\mathrm{d}t}$ | **M1** | |
| | Obtain answer $\frac{2}{3(2+3t)}$ | **A1** | OE. Express as a simple fraction but not necessarily fully cancelled. |
| | Explain why this is always positive | **A1** | For correct gradient. e.g. $x$ is only defined for $2+3t>0$ hence gradient > 0 |
| | **Alternative method for Question 6(a)** | | |
| | Form equation in $x$ and $y$ only | **M1** | |
| | Obtain $y = \frac{\mathrm{e}^x - 2}{3\mathrm{e}^x}\left(= \frac{1}{3} - \frac{2}{3}\mathrm{e}^{-x}\right)$ | **A1** | OE |
| | Differentiate | **M1** | |
| | Obtain $y' = \frac{2}{3}\mathrm{e}^{-x}$ | **A1** | OE |
| | Explain why this is always positive | **A1** | |
| | | **5** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 6(b) | Obtain $y = -\frac{1}{3}$ when $x = 0$ | **B1** | |
| | Use a correct method to form the given tangent | **M1** | $\left(\frac{y+\frac{1}{3}}{x} = \frac{2}{3}\right)$ |
| | Obtain answer $3y = 2x - 1$ | **A1** | OE |
| | | **3** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 7(a) | Use correct quotient rule or correct product rule | **M1** | e.g. $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sqrt{x}.\frac{1}{1+x^2} - \tan^{-1}x.\frac{1}{2\sqrt{x}}}{x}$ |
| | Obtain correct derivative in any form | **A1** | |
| | Equate derivative to zero and remove inverse tangent | **M1** | |
| | Obtain $a = \tan\left(\frac{2a}{1+a^2}\right)$ from correct working | **A1** | AG. Accept with $x$ in place of $a$. |
| | | **4** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 7(b) | Calculate the value of a relevant expression or pair of expressions at $a = 1.3$ and $a = 1.5$ | **M1** | Must be using radians |
| | Complete the argument correctly with correct calculated values | **A1** | e.g. $1.3 < 1.448, \quad 1.5 > 1.322$ $(0.148, -0.178)$ |
| | | **2** | |
| 7(c) | Use the iterative process $a_{n+1} = \tan\left(\frac{2a_n}{1+a_n^2}\right)$ correctly at least twice | **M1** | |
| | Obtain final answer 1.39 | **A1** | |
| | Show sufficient iterations to at least 4 d.p. to justify 1.39 to 2 d.p. or show there is a sign change in the interval (1.385, 1.395) | **A1** | Allow recovery |
| | | **3** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 8(a) | State or imply $\overrightarrow{AB} = \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix}$ | **B1** | OE. Allow $\pm$ |
| | Use the correct process to calculate the scalar product of a pair of relevant vectors, e.g. their $\overrightarrow{AB}$ and a direction vector for $l$ | **M1** | $(2 + 2 - 3 = 1)$ |
| | Using the correct process for the moduli, divide the scalar product by the product of the moduli of the two vectors and evaluate the inverse cosine of the result | **M1** | $\cos^{-1}\left(\frac{1}{\sqrt{6}\sqrt{14}}\right)$ |
| | Obtain answer $83.7°$ or 1.46 radians | **A1** | Or answers rounding to $83.7°$ or 1.46 radians |
| | | **4** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 8(b) | State or imply $\pm \overrightarrow{AP}$ and $\pm \overrightarrow{BP}$ in component form, i.e. $(1 + \lambda, 1 – 2\lambda, \lambda)$ and $(– 1 + \lambda, 2 – 2\lambda, 3 + \lambda)$, or equivalent | **B1** | |
| | Form an equation in $\lambda$ by equating moduli or by using $\cos BAP = \cos ABP$ | ***M1** | |
| | Obtain a correct equation in any form $(1+\lambda)^2+(1-2\lambda)^2+\lambda^2=(\lambda-1)^2+(2-2\lambda)^2+(\lambda+3)^2$ | **A1** | Or $(1+\lambda)\sqrt{14-4\lambda+6\lambda^2}=(13-\lambda)\sqrt{2-2\lambda+6\lambda^2}$ $\left(83\lambda^3-528\lambda^2+207\lambda-162=0\right)$ |
| | Solve for $\lambda$ and obtain position vector | **DM1** | [$\lambda = 6$] |
| | Obtain correct position vector for $P$ in any form, e.g. (8, – 9, 7) or $8\mathbf{i} – 9\mathbf{j} + 7\mathbf{k}$ | **A1** | Accept coordinates |
| | | **5** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 9(a) | Use correct product rule or correct quotient rule | **M1** | |
| | Obtain correct derivative in any form | **A1** | $y'=\frac{x^{-\frac{2}{3}}}{x}-\frac{2}{3}x^{-\frac{5}{3}}\ln x$ |
| | Equate 2 term derivative to zero and solve for $x$ | **M1** | |
| | Obtain answer $x=e^{\frac{3}{2}}$ | **A1** | Or exact equivalent |
| | Obtain answer $y=\frac{3}{2e}$ | **A1** | Or exact equivalent |
| | | **5** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 9(b) | Commence integration and reach $ax^{\frac{1}{3}}\ln x + b\int x^{\frac{1}{3}}.\frac{1}{x}\mathrm{d}x$ | ***M1** | |
| | Obtain $3x^{\frac{1}{3}}\ln x - 3\int x^{\frac{1}{3}}.\frac{1}{x}\mathrm{d}x$ | **A1** | |
| | Complete the integration and obtain $3x^{\frac{1}{3}}\ln x - 9x^{\frac{1}{3}}$ | **A1** | OE |
| | Use limits correctly in an expression of the form $px^{\frac{1}{3}}\ln x + qx^{\frac{1}{3}}$ $(pq \neq 0)$ | **DM1** | $6\ln 8 - 9\times 2 - 0 + 9$ |
| | Obtain $18\ln 2 - 9$ from full and correct working | **A1** | AG<br>need to see $\ln 8 = 3\ln 2$ |
| | | **5** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 10 | State a suitable form of partial fractions for $\frac{1}{x^2(1+2x)}$ | **B1** | e.g. $\frac{A}{x}+\frac{B}{x^2}+\frac{C}{1+2x}$ or $\frac{Ax+B}{x^2}+\frac{C}{1+2x}$ |
| | Use a relevant method to determine a constant | **M1** | |
| | Obtain one of $A=-2$, $B=1$ and $C=4$ | **A1** | |
| | Obtain a second value | **A1** | |
| | Obtain the third value | **A1** | |
| | Separate variables correctly and integrate at least one term | **M1** | |
| | Obtain terms $-2\ln x-\frac{1}{x}+2\ln(1+2x)$ and $t$ | **B3 FT** | The FT is on $A$, $B$ and $C$.<br>Withhold B1 for each error or omission. |
| | Evaluate a constant, or use limits $x=1$, $t=0$ in a solution containing terms $t$, $a\ln x$ and $b\ln(1+2x)$, where $ab\neq 0$ | **M1** | |
| | Obtain a correct expression for $t$ in any form, e.g. $t=-\frac{1}{x}+2\ln\left(\frac{1+2x}{3x}\right)+1$ | **A1** | |
| | | **11** | |

# 3 2021 | May/Jun | Variant 2 | 9709_s21_qp_32

**Topic & Sub-topic Index of Questions**

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Cambridge Assessment International Education

# Cambridge International AS & A Level

CANDIDATE NAME

CENTRE NUMBER

CANDIDATE NUMBER

*4535261400*

**MATHEMATICS** **9709/32**

Paper 3 Pure Mathematics 3 **May/June 2021**

**1 hour 50 minutes**

You must answer on the question paper.

You will need: List of formulae (MF19)

**INSTRUCTIONS**

- Answer **all** questions.
- Use a black or dark blue pen. You may use an HB pencil for any diagrams or graphs.
- Write your name, centre number and candidate number in the boxes at the top of the page.
- Write your answer to each question in the space provided.
- Do **not** use an erasable pen or correction fluid.
- Do **not** write on any bar codes.
- If additional space is needed, you should use the lined page at the end of this booklet; the question number or numbers must be clearly shown.
- You should use a calculator where appropriate.
- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

**INFORMATION**

- The total mark for this paper is 75.
- The number of marks for each question or part question is shown in brackets [ ].

This document has **20** pages. Any blank pages are indicated.

**1** Solve the inequality $|2x - 1| < 3|x + 1|$. [4]

**2** On a sketch of an Argand diagram, shade the region whose points represent complex numbers $z$ satisfying the inequalities $|z + 1 - \mathrm{i}| \leqslant 1$ and $\arg(z - 1) \leqslant \frac{3}{4}\pi$. [4]

**3** The variables $x$ and $y$ satisfy the equation $x = A(3^{-y})$, where $A$ is a constant.

**(a)** Explain why the graph of $y$ against $\ln x$ is a straight line and state the exact value of the gradient of the line. [3]

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It is given that the line intersects the $y$-axis at the point where $y = 1.3$.

**(b)** Calculate the value of $A$, giving your answer correct to 2 decimal places. [2]

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**4** Using integration by parts, find the exact value of $\int_0^2 \tan^{-1}\left(\frac{1}{2}x\right) \mathrm{d}x$. [5]

**5** The complex number $u$ is given by $u = 10 - 4\sqrt{6}\mathrm{i}$.

Find the two square roots of $u$, giving your answers in the form $a + \mathrm{i}b$, where $a$ and $b$ are real and exact. [5]

**6** **(a)** Prove that $\operatorname{cosec} 2\theta - \cot 2\theta \equiv \tan \theta$. [3]

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**(b)** Hence show that $\int_{\frac{1}{4}\pi}^{\frac{1}{3}\pi} (\operatorname{cosec} 2\theta - \cot 2\theta)\, \mathrm{d}\theta = \frac{1}{2} \ln 2$. [4]

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**7** A curve is such that the gradient at a general point with coordinates $(x, y)$ is proportional to $\frac{y}{\sqrt{x+1}}$. The curve passes through the points with coordinates (0, 1) and (3, e).

By setting up and solving a differential equation, find the equation of the curve, expressing $y$ in terms of $x$. [7]

**8** The equation of a curve is $y = e^{-5x} \tan^2 x$ for $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$.

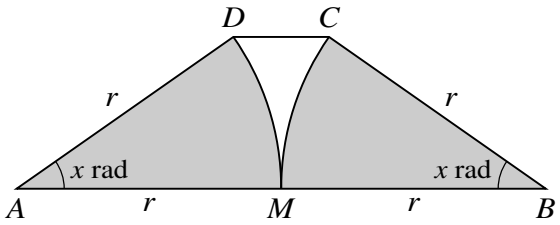
Find the $x$-coordinates of the stationary points of the curve. Give your answers correct to 3 decimal places where appropriate. [8]

**9** Let $\mathrm{f}(x) = \dfrac{14 - 3x + 2x^2}{(2 + x)(3 + x^2)}$.

**(a)** Express $\mathrm{f}(x)$ in partial fractions. [5]

**(b)** Hence obtain the expansion of $f(x)$ in ascending powers of $x$, up to and including the term in $x^2$. [5]

**10**




The diagram shows a trapezium $ABCD$ in which $AD = BC = r$ and $AB = 2r$. The acute angles $BAD$ and $ABC$ are both equal to $x$ radians. Circular arcs of radius $r$ with centres $A$ and $B$ meet at $M$, the midpoint of $AB$.

**(a)** Given that the sum of the areas of the shaded sectors is 90% of the area of the trapezium, show that $x$ satisfies the equation $x = 0.9(2 - \cos x)\sin x$. [3]

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**(b)** Verify by calculation that $x$ lies between 0.5 and 0.7. [2]

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**(c)** Show that if a sequence of values in the interval $0 < x < \frac{1}{2}\pi$ given by the iterative formula

$$x_{n+1} = \cos^{-1}\left(2 - \frac{x_n}{0.9 \sin x_n}\right)$$

converges, then it converges to the root of the equation in part **(a)**. [2]

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**(d)** Use this iterative formula to determine $x$ correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

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**11** With respect to the origin $O$, the points $A$ and $B$ have position vectors given by $\overrightarrow{OA} = 2\mathbf{i} - \mathbf{j}$ and $\overrightarrow{OB} = \mathbf{j} - 2\mathbf{k}$.

**(a)** Show that $OA = OB$ and use a scalar product to calculate angle $AOB$ in degrees. [4]

The midpoint of $AB$ is $M$. The point $P$ on the line through $O$ and $M$ is such that $PA : OA = \sqrt{7} : 1$.

**(b)** Find the possible position vectors of $P$. [6]

## Additional Page

If you use the following lined page to complete the answer(s) to any question(s), the question number(s) must be clearly shown.

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Cambridge Assessment International Education

# Cambridge International A Level

**MATHEMATICS** **9709/32**

Paper 3 Pure Mathematics 3 **May/June 2021**

MARK SCHEME

Maximum Mark: 75

**Published**

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

Cambridge International will not enter into discussions about these mark schemes.

Cambridge International is publishing the mark schemes for the May/June 2021 series for most Cambridge IGCSE™, Cambridge International A and AS Level components and some Cambridge O Level components.





**[Turn over**

**Generic Marking Principles**

These general marking principles must be applied by all examiners when marking candidate answers. They should be applied alongside the specific content of the mark scheme or generic level descriptors for a question. Each question paper and mark scheme will also comply with these marking principles.

GENERIC MARKING PRINCIPLE 1:
Marks must be awarded in line with:

- the specific content of the mark scheme or the generic level descriptors for the question
- the specific skills defined in the mark scheme or in the generic level descriptors for the question
- the standard of response required by a candidate as exemplified by the standardisation scripts.

GENERIC MARKING PRINCIPLE 2:
Marks awarded are always **whole marks** (not half marks, or other fractions).

GENERIC MARKING PRINCIPLE 3:
Marks must be awarded **positively**:

- marks are awarded for correct/valid answers, as defined in the mark scheme. However, credit is given for valid answers which go beyond the scope of the syllabus and mark scheme, referring to your Team Leader as appropriate
- marks are awarded when candidates clearly demonstrate what they know and can do
- marks are not deducted for errors
- marks are not deducted for omissions
- answers should only be judged on the quality of spelling, punctuation and grammar when these features are specifically assessed by the question as indicated by the mark scheme. The meaning, however, should be unambiguous.

GENERIC MARKING PRINCIPLE 4:
Rules must be applied consistently, e.g. in situations where candidates have not followed instructions or in the application of generic level descriptors.

GENERIC MARKING PRINCIPLE 5:
Marks should be awarded using the full range of marks defined in the mark scheme for the question (however; the use of the full mark range may be limited according to the quality of the candidate responses seen).

GENERIC MARKING PRINCIPLE 6:
Marks awarded are based solely on the requirements as defined in the mark scheme. Marks should not be awarded with grade thresholds or grade descriptors in mind.

| Mathematics Specific Marking Principles | |
|---|---|
| 1 | Unless a particular method has been specified in the question, full marks may be awarded for any correct method. However, if a calculation is required then no marks will be awarded for a scale drawing. |
| 2 | Unless specified in the question, answers may be given as fractions, decimals or in standard form. Ignore superfluous zeros, provided that the degree of accuracy is not affected. |
| 3 | Allow alternative conventions for notation if used consistently throughout the paper, e.g. commas being used as decimal points. |
| 4 | Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored (isw). |
| 5 | Where a candidate has misread a number in the question and used that value consistently throughout, provided that number does not alter the difficulty or the method required, award all marks earned and deduct just 1 mark for the misread. |
| 6 | Recovery within working is allowed, e.g. a notation error in the working where the following line of working makes the candidate's intent clear. |

**Mark Scheme Notes**

The following notes are intended to aid interpretation of mark schemes in general, but individual mark schemes may include marks awarded for specific reasons outside the scope of these notes.

**Types of mark**

**M** Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.

**A** Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).

**B** Mark for a correct result or statement independent of method marks.

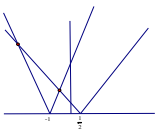
**DM** or **DB** When a part of a question has two or more 'method' steps, the M marks are generally independent unless the scheme specifically says otherwise; and similarly, when there are several B marks allocated. The notation DM or DB is used to indicate that a particular M or B mark is dependent on an earlier M or B (asterisked) mark in the scheme. When two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.

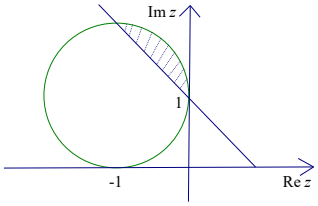
**FT** Implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A or B marks are given for correct work only.

- A or B marks are given for correct work only (not for results obtained from incorrect working) unless follow through is allowed (see abbreviation FT above).
- For a numerical answer, allow the A or B mark if the answer is correct to 3 significant figures or would be correct to 3 significant figures if rounded (1 decimal place for angles in degrees).
- The total number of marks available for each question is shown at the bottom of the Marks column.
- Wrong or missing units in an answer should not result in loss of marks unless the guidance indicates otherwise.
- Square brackets [ ] around text or numbers show extra information not needed for the mark to be awarded.

**Abbreviations**

| | |
|---|---|
| AEF/OE | Any Equivalent Form (of answer is equally acceptable) / Or Equivalent |
| AG | Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid) |
| CAO | Correct Answer Only (emphasising that no 'follow through' from a previous error is allowed) |
| CWO | Correct Working Only |
| ISW | Ignore Subsequent Working |
| SOI | Seen Or Implied |
| SC | Special Case (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance) |
| WWW | Without Wrong Working |
| AWRT | Answer Which Rounds To |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 1 | State or imply non-modular inequality $(2x-1)^2 < 3^2(x+1)^2$, or corresponding quadratic equation | **B1** | e.g. $5x^2 + 22x + 8 = 0$<br>Allow recovery from 'invisible brackets' on RHS |
| | Form and solve a 3-term quadratic in $x$ | **M1** | |
| | Obtain critical values $x = -4$ and $x = -\frac{2}{5}$ | **A1** | |
| | State final answer $x < -4,\ x > -\frac{2}{5}$ | **A1** | Do not condone $\leqslant$ for $<$, or $\geqslant$ for $>$ in the final answer.<br>Allow 'or' but not 'and'.<br>$-\frac{2}{5} < x < -4$ scores A0.<br>Accept equivalent forms using brackets e.g. $x \in (-\infty, -4) \cup (-0.4, \infty)$ |
| | **Alternative method for Question 1** | | |
| | Obtain critical value $x = -4$ from a graphical method, or by solving a linear equation or linear inequality | **B1** |  |
| | Obtain critical value $x = -\frac{2}{5}$ similarly | **B2** | |
| | State final answer $x < -4,\ x > -\frac{2}{5}$ | **B1** | Do not condone $\leqslant$ for $<$, or $\geqslant$ for $>$ in the final answer.<br>Allow 'or' but not 'and'.<br>$-\frac{2}{5} < x < -4$ scores A0.<br>Accept equivalent forms using brackets e.g. $x \in (-\infty, -4) \cup (-0.4, \infty)$ |
| | | **4** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 2 | Show a circle with centre – 1 + i. | **B1** | Need some indication of scale or a correct label. Could just be mark(s) on the axes |
| | Show a circle with radius 1 and centre not at the origin (or relevant part thereof). | **B1** | |
| | Show correct half line from 1(or relevant part thereof) . | **B1** | |
| | Shade the correct region on a correct diagram. | **B1** | |
| |  | **4** | N.B. If they have very different scales on *their* 2 axes the diagram must match *their* scale - the 'circle' should be an ellipse.<br>Allow freehand diagrams with clear correct intention. |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 3(a) | State or imply $\ln x = \ln A - y\ln 3$ | **B1** | $\left(y = -\frac{1}{\ln 3}\ln x + \frac{\ln A}{\ln 3}\right)$ |
| | State that the graph of $y$ against $\ln x$ has an equation that is *linear* in $y$ and $\ln x$, or has an equation of the standard form '$y = mx + c$' and is thus a straight line | **B1** | Must be a correct statement.<br>Accept if the 2 equations are written side by side with no comment. An equation with $y\ln 3$ should be compared with the form $py + q\ln x = c$. |
| | State that the gradient is $-\frac{1}{\ln 3}$ | **B1** | OE. Exact answer required.<br>ISW after a correct statement. |
| | | **3** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 3(b) | Substitute $\ln x = 0$, $y = 1.3$ and use correct method to solve for $A$ | **M1** | $(\ln A = 1.3\ln 3)$<br>Follow *their* equation in $y$ and $\ln x$.<br>Must be substituting $\ln x = 0$, not $x = 0$.<br>$\ln 0$ 'used' in the solution scores M0A0. |
| | Obtain answer $A = 4.17$ only | **A1** | Must be 2 d.p. as specified in question |
| | | **2** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 4 | Commence integration and reach $ax\tan^{-1}\frac{1}{2}x + b\int x\frac{1}{c+x^2}\,\mathrm{d}x$ | ***M1** | OE. Denominator might be $1+\frac{x^2}{4}$ or $2+\frac{x^2}{2}$. |
| | Obtain $x\tan^{-1}\left(\frac{1}{2}x\right) - \int x.\ \frac{2}{4+x^2}\ \mathrm{d}x$ | **A1** | OE |
| | Complete integration and obtain $x\tan^{-1}\left(\frac{1}{2}x\right) - \ln\left(4+x^2\right)$ | **A1** | OE e.g. with $\ln\left(1+\frac{x^2}{4}\right)$ |
| | Substitute limits correctly in an expression of the form $px\tan^{-1}x + q\ln\left(c+x^2\right)$ | **DM1** | $2\tan^{-1}1 - \ln 8 + \ln 4$ OE |
| | Obtain final answer $\frac{1}{2}\pi - \ln 2$ | **A1** | OE exact answer.<br>Needs a value for $\tan^{-1}1$ and a single log term |
| | **Alternative method for Question 4** | | |
| | Use the substitution $\theta = \tan^{-1}\frac{x}{2}$ to obtain $\lambda\int 2\theta\sec^2\theta\,\mathrm{d}\theta$ and reach $p\theta\tan\theta + q\int\tan\theta\,\mathrm{d}\theta$ | ***M1** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 4 | Obtain $2\theta\tan\theta - 2\int\tan\theta\,\mathrm{d}\theta$ | **A1** | OE |
| | Complete integration and obtain $2\theta\tan\theta + 2\ln(\cos\theta)$ | **A1** | OE |
| | Substitute correct limits correctly in an expression of the form $r\theta\tan\theta + s\ln(\cos\theta)$ | **DM1** | Limits should be $\frac{\pi}{4}$ and 0. Limits must be in radians. |
| | Obtain final answer $\frac{1}{2}\pi - \ln 2$ | **A1** | OE exact answer.<br>Need values for trig. functions and a single log term. |
| | | **5** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 5 | Square $a + \mathrm{i}b$, use $\mathrm{i}^2 = -1$ and equate real and imaginary parts to 10 and $-4\sqrt{6}$ respectively | **M1** | |
| | Obtain $a^2 - b^2 = 10$ and $2ab = -4\sqrt{6}$ | **A1** | Allow $2ab\mathrm{i} = -4\sqrt{6}\mathrm{i}$ |
| | Eliminate one unknown and find an equation in the other | **M1** | Must be sensible algebra e.g. use of $\sqrt{a^2 - b^2} = a - b$ socres M0 |
| | Obtain $a^4 - 10a^2 - 24[= 0]$, or $b^4 + 10b^2 - 24[= 0]$, or 3-term equivalent | **A1** | Or equivalent horizontal equation from correct work |
| | Obtain final answers $\pm\left(2\sqrt{3} - \sqrt{2}\mathrm{i}\right)$, or exact equivalents | **A1** | e.g. $\pm\left(\sqrt{12} - \sqrt{2}\mathrm{i}\right)$ from correct work |
| | **Alternative method for Question 5** | | |
| | Use the correct method to find the modulus and argument of $\sqrt{u}$ | **M1** | |
| | Obtain modulus $\sqrt{14}$ | **A1** | |
| | Obtain argument $\tan^{-1}\frac{-1}{\sqrt{6}}$ using an exact method | **A1** | e.g. by using half angle formula which gives $2\sqrt{6}t^2 - 10t - 2\sqrt{6} = 0$ |
| | Convert to the required form | **M1** | $\pm\sqrt{14}\left(\frac{\sqrt{6}}{\sqrt{7}} - \frac{1}{\sqrt{7}}\mathrm{i}\right)$<br>This mark is available if working in decimals |
| | Obtain answers $\pm\left(2\sqrt{3} - \sqrt{2}\mathrm{i}\right)$, or exact equivalents | **A1** | e.g. $\pm\left(\sqrt{12} - \sqrt{2}\mathrm{i}\right)$ |
| | | **5** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 6(a) | Express the LHS in terms of $\cos 2\theta$ and $\sin 2\theta$ | **B1** | e.g. $\frac{1}{\sin 2\theta} - \frac{\cos 2\theta}{\sin 2\theta}$ |
| | Use correct double angle formulae to express the LHS in terms of $\cos\theta$ and $\sin\theta$ | **M1** | e.g. $\frac{1-\left(1-2\sin^2\theta\right)}{2\sin\theta\cos\theta}$ |
| | Obtain $\tan\theta$ from correct working | **A1** | AG |
| | **Alternative method for Question 6(a)** | | |
| | Express the LHS in terms of $\sin 2\theta$ and $\tan 2\theta$ | **B1** | |
| | Use correct double angle formulae to express the LHS in terms of $\cos\theta$ and $\sin\theta$ | **M1** | e.g. $\frac{1}{2\sin\theta\cos\theta} - \frac{1-\frac{\sin^2\theta}{\cos^2\theta}}{2\frac{\sin\theta}{\cos\theta}}\left(=\frac{4\sin^2\theta}{4\sin\theta\cos\theta}\right)$ |
| | Obtain $\tan\theta$ from correct working | **A1** | AG |
| | **Alternative method for Question 6(a)** | | |
| | Express the LHS in terms of $\sin 2\theta$ and $\tan 2\theta$ | **B1** | |
| | Use correct $t$ substitution or rearrangement of $\sin 2\theta$ in terms of $\sec^2 2\theta$ and $\tan\theta$ to express the LHS in terms of $\tan\theta$. | **M1** | $\left(\frac{\sec^2\theta}{2\tan\theta} - \frac{1-\tan^2\theta}{2\tan\theta} =\right)\frac{1+\tan^2}{2\tan} - \frac{1-\tan^2}{2\tan}$ |
| | Obtain $\tan\theta$ from correct working | **A1** | AG |
| | | **3** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 6(b) | State integral of the form $\mp\ln\cos\theta$ or $\pm\ln\sec\theta$ | ***M1** | $\left[-\ln\cos\theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$ OE |
| | Use correct limits correctly and insert exact values for the trigonometric ratios | **DM1** | Need to see evidence of the substitution |
| | Obtain a correct expression, e.g. $-\ln\frac{1}{2}+\ln\frac{1}{\sqrt{2}}$ | **A1** | |
| | Obtain $\frac{1}{2}\ln 2$ from correct working | **A1** | AG (must see an intermediate step) |
| | | **4** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 7 | State equation $\frac{\mathrm{d}y}{\mathrm{d}x} = k\frac{y}{\sqrt{x+1}}$ | **B1** | OE. Must be a differential equation. |
| | Separate variables correctly for *their* differential equation and integrate at least one side | ***M1** | $\int \frac{1}{y}\mathrm{d}y = \int \frac{k}{\sqrt{x+1}}\mathrm{d}x$ |
| | Obtain $\ln y$ | **A1** | Allow M1A1A1 if they have assumed $k = 1$ or are working with an incorrect value for $k$ |
| | Obtain $2[k]\sqrt{x+1}$ | **A1** | |
| | Use (0, 1) **and** (3, e) in an expression containing $\ln y$ and $\sqrt{x+1}$ and a constant of integration to determine $k$ and/or a constant of integration $c$ (or use (0, 1), (3, e) and $(x, y)$ as limits on definite integrals) | **DM1** | If remove logs before finding the constant of integration then the constant must be of the correct form. |
| | Obtain $k = \frac{1}{2}$ and $c = -1$ | **A1** | OE. $\left(\ln y = \sqrt{x+1} - 1\right)$<br>Their value of $c$ will depend on where $c$ is in their equation and whether they are working with $\frac{1}{k}\ln y$.<br>The value of $k$ must be consistent with what they integrated. |
| | Obtain $y = \exp\left(\sqrt{x+1} - 1\right)$ | **A1** | NFWW, OE, ISW. |
| | | **7** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 8 | Use correct product (or quotient) rule | **M1** | At least 3 of 4 terms correct |
| | Obtain $\frac{\mathrm{d}y}{\mathrm{d}x} = -5\mathrm{e}^{-5x}\tan^2 x + 2\mathrm{e}^{-5x}\tan x\sec^2 x$ | **A1** | OE. |
| | Equate *their* derivative to zero, use $\sec^2 x = 1 + \tan^2 x$ and obtain an equation in $\tan x$ | **M1** | |
| | Obtain $2\tan^2 x - 5\tan x + 2 = 0$ | **A1** | Allow $2\tan^3 x - 5\tan^2 x + 2\tan x = 0$ |
| | State answer $x = 0$ | **B1** | From correct derivative. |
| | Solve a 3 term quadratic in $\tan x$ and obtain a value of $x$ | **M1** | Must be in radians |
| | Obtain answer, e.g. 0.464 | **A1** | Must be 3 d.p. as specified in the question.<br>Allow A1A0 if both values given to 2 d.p. or > 3 d.p. |
| | Obtain second non-zero answer, e.g. 1.107 and no other in the given interval | **A1** | |
| | **Alternative method for Question 8** | | |
| | Use correct product (or quotient) rule | **M1** | At least 3 of 4 terms correct |
| | Obtain $\frac{\mathrm{d}y}{\mathrm{d}x} = -5\mathrm{e}^{-5x}\tan^2 x + 2\mathrm{e}^{-5x}\tan x\sec^2 x$ | **A1** | OE |
| | Equate *their* derivative to zero and obtain an equation in $\sin x$ and $\cos x$ | **M1** | |
| | Obtain $5\cos x \sin x = 2$ | **A1** | Or simplified equivalent (i.e. cancelled) |
| | State answer $x = 0$ | **B1** | From correct derivative. |
| | Use double angle formula or square both sides and solve for $x$ | **M1** | Or equivalent method. Must be in radians. |
| | Obtain answer, e.g. 0.464 | **A1** | Must be 3 d.p. as specified in the question.<br>Allow A1A0 if both values given to 2 d.p. or > 3 d.p. |
| | Obtain second non-zero answer, e.g. 1.107 and no other in the given interval | **A1** | |
| | | **8** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 9(a) | State or imply the form $\frac{A}{2+x} + \frac{B+Cx}{3+x^2}$ | **B1** | |
| | Use a correct method for finding a constant | **M1** | SOI |
| | Obtain one of $A = 4$, $B = 1$ and $C = -2$ | **A1** | |
| | Obtain a second value | **A1** | |
| | Obtain the third value | **A1** | ISW |
| | | **5** | |
| 9(b) | Use correct method to find the first two terms of the expansion of $(2+x)^{-1}$, $\left(1+\frac{1}{2}x\right)^{-1}$, $(3+x^2)^{-1}$ or $\left(1+\frac{1}{3}x^2\right)^{-1}$ | **M1** | Allow unsimplified but not if still including ${}^nC_r$ |
| | Obtain correct unsimplified expansions up to the term in $x^2$ of each partial fraction | **A1 FT**<br>**A1 FT** | $2\left(1-\frac{1}{2}x+\left(\frac{1}{2}x\right)^2....\right)$<br>$+\frac{1}{3}(1-2x)\left(1-\frac{1}{3}x^2....\right)$<br>The FT is on *their A*, *B* and *C* |
| | Multiply out, up to the terms in $x^2$, by $B + Cx$, where $BC \neq 0$ | **M1** | Allow with *B* and *C* as implied in part **(b)** |
| | Obtain final answer $\frac{7}{3}-\frac{5}{3}x+\frac{7}{18}x^2$ | **A1** | Or equivalent in form $p+qx+rx^2$.<br>A0 if they multiply through by 18. |
| | | **5** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 10(a) | State or imply $CD = 2r - 2r\cos x$ | **B1** | |
| | Using correct formulae for area of sector and trapezium, or equivalent, form an equation in $r$ and $x$ | **M1** | e.g. $2\times\frac{1}{2}r^2x = \frac{0.9}{2}(2r + 2r - 2r\cos x)r\sin x$ |
| | Obtain $x = 0.9(2 - \cos x)\sin x$ | **A1** | AG, NFWW |
| | | **3** | |
| 10(b) | Calculate the values of a relevant expression or pair of expressions at $x = 0.5$ and $x = 0.7$ | **M1** | Calculated for both values and correct for one value is sufficient for M1. Must be working in radians. |
| | Complete the argument correctly with correct values | **A1** | Must have sufficient accuracy to support the answer<br>e.g. $\begin{matrix}0.5 > 0.484\\ 0.7 < 0.716\end{matrix}$ or $\begin{matrix}0.016 > 0\\ -0.016 < 0\end{matrix}$ or $\begin{matrix}0.96... < 1\\ 1.02... > 1\end{matrix}$ |
| | | **2** | |
| 10(c) | State a suitable equation, e.g. $\cos x = \left(2 - \frac{x}{0.9\sin x}\right)$ | **B1** | If working in reverse, the first B1 is for $\frac{x}{0.9\sin x} = 2 - \cos x$ |
| | Rearrange this as $x = 0.9\sin x(2 - \cos x)$ | **B1** | Need to see the complete sequence of changes. |
| | | **2** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 10(d) | Use the iterative process correctly at least once | **M1** | Must be working in radians |
| | Obtain answer 0.62 | **A1** | |
| | Show sufficient iterations to at least 4 d.p.to justify 0.62 to 2 d.p., or show there is a sign change in the interval (0.615, 0.625) | **A1** | Allow recovery.<br>N.B. A candidate who starts with 0.5 and stops at 0.61 or starts at 0.7 and stops at 0.63 can score M1A0A1 if they have worked to the required accuracy. |
| | | **3** | |
| 11(a) | Show that $OA = OB = \sqrt{5}$ | **B1** | CWO |
| | Evaluate the scalar product of the correct position vectors | **M1** | e.g. $(0 - 1 + 0)$<br>Condone of using *AO* and/or *BO* |
| | Divide *their* scalar product by the product of the moduli of *their* vectors and evaluate the inverse cosine of the result | **M1** | Much reach an angle. The question asks for the use of scalar product, so alternative methods (e.g. cosine rule) are not accepted. |
| | Obtain answer 101.5° | **A1** | The question asks for an answer in degrees. Accept 102° or better. Mark radians (1.77) as a misread.<br>Do not ISW: 78.5° as final answer scores A0. |
| | | **4** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 11(b) | State or imply $M$ has position vector $\mathbf{i} - \mathbf{k}$ | **B1** | OE |
| | Taking a general point of $OM$ to have position vector $\lambda\mathbf{i} - \lambda\mathbf{k}$, express $AP = \sqrt{7}\ OA$ as an equation in $\lambda$ | ***M1** | $\lambda\left(their\ \overrightarrow{OM}\right)$ |
| | State a correct equation in any form | **A1** | e.g. $\sqrt{(-2+\lambda)^2+1+(-\lambda)^2} = \sqrt{7}\sqrt{5}$ |
| | Reduce to $\lambda^2 - 2\lambda - 15 = 0$ | **A1** | OE |
| | Solve a quadratic and state a position vector | **DM1** | |
| | Obtain answers $5\mathbf{i} - 5\mathbf{k}$ and $-3\mathbf{i} + 3\mathbf{k}$ | **A1** | Accept coordinates |
| | **Alternative method for Question 11(b)** | | |
| | State or imply that $OP = \gamma\sqrt{2}$ | **B1** | |
| | State or imply that $\cos\frac{1}{2}AOB = \sqrt{\frac{2}{5}}$ and use cosine rule to form an equation in $\gamma$ | ***M1** | Allow $\cos\frac{1}{2}AOB = 0.632...$ |
| | State a correct equation in any form | **A1** | e.g. $35 = 5 + 2\gamma^2 - 2\sqrt{5}.\gamma\sqrt{2}.\frac{\sqrt{2}}{\sqrt{5}}$ |
| | Reduce to $\gamma^2 - 2\gamma - 15 = 0$ | **A1** | OE |
| | Solve a quadratic and state a position vector | **DM1** | |
| | Obtain answers $5\mathbf{i} - 5\mathbf{k}$ and $-3\mathbf{i} + 3\mathbf{k}$ | **A1** | Accept coordinates |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 11(b) | **Alternative method for Question 11(b)** | | |
| | State or imply $M$ has position vector $\mathbf{i} - \mathbf{k}$ | **B1** | OE |
| | State or imply that $AM = \sqrt{3}$ | **B1** | |
| | Use Pythagoras to find $MP$ | ***M1** | $MP = \sqrt{35 - (AM)^2}$ |
| | Obtain $MP = 4\sqrt{2}$ | **A1** | |
| | Correct method to find a position vector | **DM1** | $(\mathbf{i} - \mathbf{k}) \pm 4(\mathbf{i} - \mathbf{k})$ |
| | Obtain answers $5\mathbf{i} - 5\mathbf{k}$ and $-3\mathbf{i} + 3\mathbf{k}$ | **A1** | Accept coordinates |
| | | **6** | |

# 4 2021 | May/Jun | Variant 3 | 9709_s21_qp_33

**Topic & Sub-topic Index of Questions**

| No. | Sub Q. | Topic | Subtopic | QP Page | MS Page |
|---|---|---|---|---|---|
| 1 | | Pure Mathematics 3 | Algebra | 121 | 146 |
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| 3 | a | Pure Mathematics 3 | Differentiation | 123 | 147 |
| 3 | b | Pure Mathematics 3 | Differentiation | 123 | 147 |
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| 5 | a | Pure Mathematics 3 | Trigonometry | 125 | 148 |
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| 6 | a | Pure Mathematics 3 | Numerical solution of equations | 127 | 149 |
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| 10 | a | Pure Mathematics 3 | Complex numbers | 135 | 153 |
| 10 | b | Pure Mathematics 3 | Complex numbers | 135 | 153 |

Cambridge Assessment International Education

# Cambridge International AS & A Level

CANDIDATE NAME

CENTRE NUMBER

CANDIDATE NUMBER

*2912457036*

**MATHEMATICS** **9709/33**

Paper 3 Pure Mathematics 3 **May/June 2021**

**1 hour 50 minutes**

You must answer on the question paper.

You will need: List of formulae (MF19)

**INSTRUCTIONS**

- Answer **all** questions.
- Use a black or dark blue pen. You may use an HB pencil for any diagrams or graphs.
- Write your name, centre number and candidate number in the boxes at the top of the page.
- Write your answer to each question in the space provided.
- Do **not** use an erasable pen or correction fluid.
- Do **not** write on any bar codes.
- If additional space is needed, you should use the lined page at the end of this booklet; the question number or numbers must be clearly shown.
- You should use a calculator where appropriate.
- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

**INFORMATION**

- The total mark for this paper is 75.
- The number of marks for each question or part question is shown in brackets [ ].

This document has **20** pages. Any blank pages are indicated.

**1** Expand $(1 + 3x)^{\frac{2}{3}}$ in ascending powers of $x$, up to and including the term in $x^3$, simplifying the coefficients. [4]

**2** Solve the equation $4^x = 3 + 4^{-x}$. Give your answer correct to 3 decimal places. [5]

**3** The parametric equations of a curve are

$$x = t + \ln(t + 2), \qquad y = (t - 1)e^{-2t},$$

where $t > -2$.

**(a)** Express $\frac{dy}{dx}$ in terms of $t$, simplifying your answer. [5]

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**(b)** Find the exact $y$-coordinate of the stationary point of the curve. [2]

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**4** Let $\mathrm{f}(x) = \dfrac{15 - 6x}{(1 + 2x)(4 - x)}$.

**(a)** Express $\mathrm{f}(x)$ in partial fractions. [3]

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**(b)** Hence find $\int_1^2 \mathrm{f}(x)\,\mathrm{d}x$, giving your answer in the form $\ln\left(\dfrac{a}{b}\right)$, where $a$ and $b$ are integers. [4]

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**5** **(a)** By first expanding $\tan(2\theta + 2\theta)$, show that the equation $\tan 4\theta = \frac{1}{2}\tan\theta$ may be expressed as $\tan^4\theta + 2\tan^2\theta - 7 = 0$. [4]

**(b)** Hence solve the equation $\tan 4\theta = \frac{1}{2}\tan\theta$, for $0° < \theta < 180°$. [3]

**6** **(a)** By sketching a suitable pair of graphs, show that the equation $\cot \frac{1}{2}x = 1 + e^{-x}$ has exactly one root in the interval $0 < x \leqslant \pi$. [2]

**(b)** Verify by calculation that this root lies between 1 and 1.5. [2]

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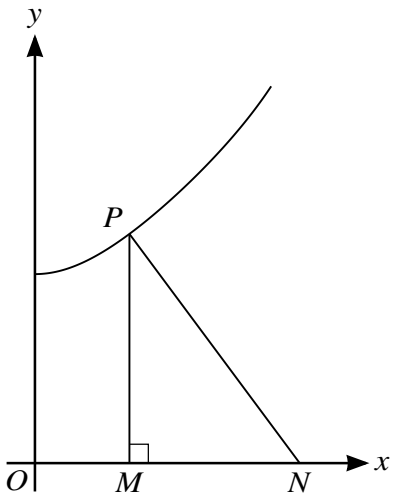
**(c)** Use the iterative formula $x_{n+1} = 2\tan^{-1}\left(\frac{1}{1+e^{-x_n}}\right)$ to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

**7**




For the curve shown in the diagram, the normal to the curve at the point $P$ with coordinates $(x, y)$ meets the $x$-axis at $N$. The point $M$ is the foot of the perpendicular from $P$ to the $x$-axis.

The curve is such that for all values of $x$ in the interval $0 \leqslant x < \frac{1}{2}\pi$, the area of triangle $PMN$ is equal to $\tan x$.

**(a)** **(i)** Show that $\dfrac{MN}{y} = \dfrac{\mathrm{d}y}{\mathrm{d}x}$. [1]

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**(ii)** Hence show that $x$ and $y$ satisfy the differential equation $\frac{1}{2}y^2 \dfrac{\mathrm{d}y}{\mathrm{d}x} = \tan x$. [2]

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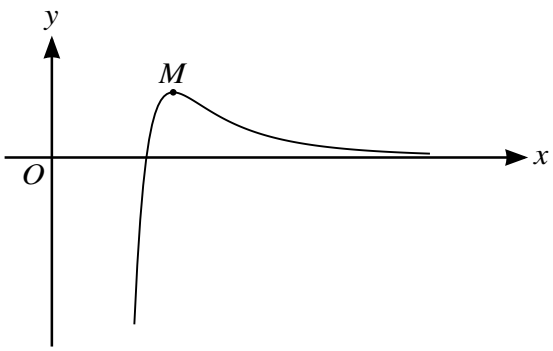
**(b)** Given that $y = 1$ when $x = 0$, solve this differential equation to find the equation of the curve, expressing $y$ in terms of $x$. [6]

**8**




The diagram shows the curve $y = \frac{\ln x}{x^4}$ and its maximum point $M$.

**(a)** Find the exact coordinates of $M$. [4]

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**(b)** By using integration by parts, show that for all $a > 1$, $\int_1^a \frac{\ln x}{x^4}\,\mathrm{d}x < \frac{1}{9}$. [6]

**9** The quadrilateral $ABCD$ is a trapezium in which $AB$ and $DC$ are parallel. With respect to the origin $O$, the position vectors of $A$, $B$ and $C$ are given by $\overrightarrow{OA} = -\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\overrightarrow{OB} = \mathbf{i} + 3\mathbf{j} + \mathbf{k}$ and $\overrightarrow{OC} = 2\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$.

**(a)** Given that $\overrightarrow{DC} = 3\overrightarrow{AB}$, find the position vector of $D$. [3]

**(b)** State a vector equation for the line through $A$ and $B$. [1]

**(c)** Find the distance between the parallel sides and hence find the area of the trapezium. [5]

**10** **(a)** Verify that $-1 + \sqrt{2}\mathrm{i}$ is a root of the equation $z^4 + 3z^2 + 2z + 12 = 0$. [3]

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**(b)** Find the other roots of this equation. [7]

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## Additional Page

If you use the following lined page to complete the answer(s) to any question(s), the question number(s) must be clearly shown.

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### 4.1 2021 | May/Jun | Variant 3 | 9709_s21_ms_33

Cambridge Assessment International Education

# Cambridge International A Level

**MATHEMATICS** **9709/33**

Paper 3 Pure Mathematics 3 **May/June 2021**

MARK SCHEME

Maximum Mark: 75

**Published**

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

Cambridge International will not enter into discussions about these mark schemes.

Cambridge International is publishing the mark schemes for the May/June 2021 series for most Cambridge IGCSE™, Cambridge International A and AS Level components and some Cambridge O Level components.





**[Turn over**

**PUBLISHED**

**Generic Marking Principles**

These general marking principles must be applied by all examiners when marking candidate answers. They should be applied alongside the specific content of the mark scheme or generic level descriptors for a question. Each question paper and mark scheme will also comply with these marking principles.

GENERIC MARKING PRINCIPLE 1:
Marks must be awarded in line with:

- the specific content of the mark scheme or the generic level descriptors for the question
- the specific skills defined in the mark scheme or in the generic level descriptors for the question
- the standard of response required by a candidate as exemplified by the standardisation scripts.

GENERIC MARKING PRINCIPLE 2:
Marks awarded are always **whole marks** (not half marks, or other fractions).

GENERIC MARKING PRINCIPLE 3:
Marks must be awarded **positively**:

- marks are awarded for correct/valid answers, as defined in the mark scheme. However, credit is given for valid answers which go beyond the scope of the syllabus and mark scheme, referring to your Team Leader as appropriate
- marks are awarded when candidates clearly demonstrate what they know and can do
- marks are not deducted for errors
- marks are not deducted for omissions
- answers should only be judged on the quality of spelling, punctuation and grammar when these features are specifically assessed by the question as indicated by the mark scheme. The meaning, however, should be unambiguous.

GENERIC MARKING PRINCIPLE 4:
Rules must be applied consistently, e.g. in situations where candidates have not followed instructions or in the application of generic level descriptors.

GENERIC MARKING PRINCIPLE 5:
Marks should be awarded using the full range of marks defined in the mark scheme for the question (however; the use of the full mark range may be limited according to the quality of the candidate responses seen).

GENERIC MARKING PRINCIPLE 6:
Marks awarded are based solely on the requirements as defined in the mark scheme. Marks should not be awarded with grade thresholds or grade descriptors in mind.

| Mathematics Specific Marking Principles | |
|---|---|
| 1 | Unless a particular method has been specified in the question, full marks may be awarded for any correct method. However, if a calculation is required then no marks will be awarded for a scale drawing. |
| 2 | Unless specified in the question, answers may be given as fractions, decimals or in standard form. Ignore superfluous zeros, provided that the degree of accuracy is not affected. |
| 3 | Allow alternative conventions for notation if used consistently throughout the paper, e.g. commas being used as decimal points. |
| 4 | Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored (isw). |
| 5 | Where a candidate has misread a number in the question and used that value consistently throughout, provided that number does not alter the difficulty or the method required, award all marks earned and deduct just 1 mark for the misread. |
| 6 | Recovery within working is allowed, e.g. a notation error in the working where the following line of working makes the candidate's intent clear. |

**PUBLISHED**

**Mark Scheme Notes**

The following notes are intended to aid interpretation of mark schemes in general, but individual mark schemes may include marks awarded for specific reasons outside the scope of these notes.

**Types of mark**

**M** Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.

**A** Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).

**B** Mark for a correct result or statement independent of method marks.

**DM** or **DB** When a part of a question has two or more 'method' steps, the M marks are generally independent unless the scheme specifically says otherwise; and similarly, when there are several B marks allocated. The notation DM or DB is used to indicate that a particular M or B mark is dependent on an earlier M or B (asterisked) mark in the scheme. When two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.

**FT** Implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A or B marks are given for correct work only.

- A or B marks are given for correct work only (not for results obtained from incorrect working) unless follow through is allowed (see abbreviation FT above).
- For a numerical answer, allow the A or B mark if the answer is correct to 3 significant figures or would be correct to 3 significant figures if rounded (1 decimal place for angles in degrees).
- The total number of marks available for each question is shown at the bottom of the Marks column.
- Wrong or missing units in an answer should not result in loss of marks unless the guidance indicates otherwise.
- Square brackets [ ] around text or numbers show extra information not needed for the mark to be awarded.

**Abbreviations**

| | |
|---|---|
| AEF/OE | Any Equivalent Form (of answer is equally acceptable) / Or Equivalent |
| AG | Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid) |
| CAO | Correct Answer Only (emphasising that no 'follow through' from a previous error is allowed) |
| CWO | Correct Working Only |
| ISW | Ignore Subsequent Working |
| SOI | Seen Or Implied |
| SC | Special Case (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance) |
| WWW | Without Wrong Working |
| AWRT | Answer Which Rounds To |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 1 | State correct first two terms $1 + 2x$ | **B1** | |
| | State a correct unsimplified version of the $x^2$ or $x^3$ term | **M1** | Symbolic binomial coefficients are not sufficient for the M mark. |
| | Obtain the next term $-x^2$ | **A1** | |
| | Obtain the final term $\frac{4}{3}x^3$ | **A1** | |
| | | **4** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 2 | State or imply $u^2 - 3u - 1 = 0$, or equivalent in $4^x$ | **B1** | |
| | Solve for $u$ or $4^x$ | **M1** | |
| | Obtain root $\frac{1}{2}\left(3+\sqrt{13}\right)$, or decimal in [3.30, 3.31] | **A1** | |
| | Use correct method for finding $x$ from a positive root | **M1** | |
| | Obtain answer $x = 0.862$ and no other | **A1** | |
| | | **5** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 3(a) | State $\frac{\mathrm{d}x}{\mathrm{d}t} = 1 + \frac{1}{t+2}$ | **B1** | |
| | Use product rule | **M1** | |
| | Obtain $\frac{\mathrm{d}y}{\mathrm{d}t} = \mathrm{e}^{-2t} - 2(t-1)\mathrm{e}^{-2t}$ | **A1** | OE |
| | Use $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \div \frac{\mathrm{d}x}{\mathrm{d}t}$ | **M1** | |
| | Obtain correct answer in any simplified form, e.g. $\frac{(3-2t)(t+2)}{t+3}\ \mathrm{e}^{-2t}$ | **A1** | |
| | | **5** | |
| 3(b) | Equate derivative to zero and solve for $t$ | **M1** | |
| | Obtain $t = \frac{3}{2}$ and obtain answer $y = \frac{1}{2}\mathrm{e}^{-3}$, or exact equivalent | **A1** | |
| | | **2** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 4(a) | State or imply the form $\frac{A}{1+2x} + \frac{B}{4-x}$ and use a correct method to find a constant | **M1** | |
| | Obtain one of $A = 4$ and $B = -1$ | **A1** | |
| | Obtain the second value | **A1** | |
| | | **3** | |
| 4(b) | Integrate and obtain terms $2\ln(1+2x) + \ln(4-x)$ | **B1FT +B1FT** | The FT is on $A$ and $B$. |
| | Substitute limits correctly in an integral of the form $a\ln(1+2x) + b\ln(4-x)$, where $ab \neq 0$ | **M1** | |
| | Obtain final answer $\ln\left(\frac{50}{27}\right)$ | **A1** | |
| | | **4** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 5(a) | Use double angle formula to express tan $4\theta$ in terms of tan $2\theta$ | **M1** | |
| | Use double angle formula to express result in terms of tan $\theta$ | **M1** | |
| | Obtain a correct equation in tan $\theta$ in any form | **A1** | |
| | Obtain the given answer | **A1** | |
| | | **4** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 5(b) | Solve for tan $\theta$ and obtain a value of $\theta$ | **M1** | |
| | Obtain answer, e.g. 53.5° | **A1** | |
| | Obtain second answer, e.g. 126.5° and no other in the interval | **A1** | Ignore answers outside the given interval. Treat answers in radians as a misread. |
| | | **3** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 6(a) | Sketch a relevant graph, e.g. $y = \cot\frac{1}{2}x$ | **B1** | |
| | Sketch a second relevant graph, e.g. $y = 1 + \mathrm{e}^{-x}$, and justify the given statement | **B1** | |
| | | **2** | |
| 6(b) | Calculate values of a relevant expression or pair of expressions at $x = 1$ and $x = 1.5$ | **M1** | |
| | Complete the argument correctly with correct calculated values | **A1** | |
| | | **2** | |
| 6(c) | Use the iterative formula correctly at least once | **M1** | |
| | Obtain final answer 1.34 | **A1** | |
| | Show sufficient iterations to 4 d.p. to justify 1.34 to 2 d.p. or show there is a sign change in the interval (1.335, 1.345) | **A1** | |
| | | **3** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 7(a)(i) | Justify the given statement $\frac{MN}{y} = \frac{\mathrm{d}y}{\mathrm{d}x}$ | **B1** | |
| | | **1** | |
| 7(a)(ii) | Express the area of $PMN$ in terms of $y$ and $\frac{\mathrm{d}y}{\mathrm{d}x}$ and equate to $\tan x$ | **M1** | |
| | Obtain the given equation correctly | **A1** | |
| | | **2** | |
| 7(b) | Separate variables and integrate at least one side | **M1** | |
| | Obtain term $\frac{1}{6}y^3$ | **A1** | |
| | Obtain term of the form $\pm\ln\cos x$ | **M1** | |
| | Evaluate a constant or use $x = 0$ and $y = 1$ in a solution containing terms $ay^3$ and $\pm\ln\cos x$, or equivalent | **M1** | |
| | Obtain correct answer in any form, e.g. $\frac{1}{6}y^3 = -\ln\cos x + \frac{1}{6}$ | **A1** | |
| | Obtain final answer $y = \sqrt[3]{(1-6\ln\cos x)}$ | **A1** | OE |
| | | **6** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 8(a) | Use quotient or product rule | **M1** | |
| | Obtain correct derivative in any form | **A1** | |
| | Equate derivative to zero and solve for $x$ | **M1** | |
| | Obtain $x = \sqrt[4]{\mathrm{e}}$ and $y = \frac{1}{4\mathrm{e}}$, or exact equivalents | **A1** | |
| | | **4** | |
| 8(b) | Commence integration and reach $ax^{-3}\ln x + b\int x^{-3}.\ \frac{1}{x}\ \mathrm{d}x$ | ***M1** | |
| | Obtain $-\frac{1}{3}x^{-3}\ln x + \frac{1}{3}\int x^{-3}.\ \frac{1}{x}\ \mathrm{d}x$ | **A1** | OE |
| | Complete integration and obtain $-\frac{1}{3}x^{-3}\ln x - \frac{1}{9}x^{-3}$ | **A1** | |
| | Substitute limits correctly, having integrated twice | **DM1** | |
| | Obtain answer $\frac{1}{9} - \frac{1}{3}a^{-3}\ln a - \frac{1}{9}a^{-3}$ | **A1** | OE |
| | Justify the given statement | **A1** | |
| | | **6** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 9(a) | State or imply $\overrightarrow{AB} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ | **B1** | OE |
| | Carry out a correct method to find $\overrightarrow{OD}$ | **M1** | |
| | Obtain answer $-4\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ | **A1** | OE |
| | | **3** | |
| 9(b) | State $\mathbf{r} = -\mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + \lambda(2\mathbf{i} + \mathbf{j} - 2\mathbf{k})$ | **B1FT** | OE. The FT is on $\overrightarrow{AB}$. |
| | | **1** | |
| 9(c) | For a general point $P$ on $AB$, state $\overrightarrow{CP}$ or $\overrightarrow{DP}$ in component form, e.g. $\overrightarrow{CP} = (3 - 2\lambda, -\lambda, -6 + 2\lambda)$ | ***M1** | |
| | Equate a relevant scalar product to zero *or* equate derivative of $\lvert\overrightarrow{CP}\rvert$ to zero *or* use Pythagoras in a relevant triangle and solve for $\lambda$ | **DM1** | |
| | Obtain $\lambda = 2$ | **A1** | |
| | Show the perpendicular is of length 3 | **A1** | |
| | Carry out a correct method to find the area of $ABCD$ and obtain the answer 18 | **A1** | |
| | **Alternative method for Question 9(c)** | | |
| | Use a scalar product to find the projection $CN$ (or $DN$) of $BC$ (or $AD$) on $CD$ | ***M1** | |
| | Obtain $CN = 3$ (or $DN = 3$) | **A1** | |
| | Use Pythagoras to obtain $BN$ (or $AN$) | **DM1** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 9(c) cont'd | Obtain answer 3 | **A1** | |
| | Carry out a correct method to find the area of *ABCD* and obtain the answer 18 | **A1** | |
| | | **5** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 10(a) | Substitute $-1+\sqrt{2}\mathrm{i}$ and attempt expansions of the $z^2$ and $z^4$ terms | **M1** | |
| | Use $\mathrm{i}^2=-1$ at least once | **M1** | |
| | Complete the verification correctly | **A1** | |
| | | **3** | |
| 10(b) | State second root $-1-\sqrt{2}\mathrm{i}$ | **A1** | |
| | Carry out a method to find a quadratic factor with zeros $-1\pm\sqrt{2}\mathrm{i}$ | **M1** | |
| | Obtain $z^2+2z+3$ | **A1** | |
| | Commence division and reach partial quotient $z^2+kz$ | **M1** | |
| | Obtain second quadratic factor $z^2-2z+4$ | **A1** | |
| | Solve a 3-term quadratic and use $\mathrm{i}^2=-1$ | **M1** | |
| | Obtain roots $1+\sqrt{3}\mathrm{i}$ and $1-\sqrt{3}\mathrm{i}$ | **A1** | |
| | | **7** | |

# 5 2021 | Oct/Nov | Variant 1 | 9709_w21_qp_31

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Cambridge Assessment International Education

# Cambridge International AS & A Level

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|---|---|
| CANDIDATE NAME | |
| CENTRE NUMBER | |
| CANDIDATE NUMBER | |

*0746866051*

**MATHEMATICS** **9709/31**

Paper 3 Pure Mathematics 3 **October/November 2021**

**1 hour 50 minutes**

You must answer on the question paper.

You will need: List of formulae (MF19)

**INSTRUCTIONS**

- Answer **all** questions.
- Use a black or dark blue pen. You may use an HB pencil for any diagrams or graphs.
- Write your name, centre number and candidate number in the boxes at the top of the page.
- Write your answer to each question in the space provided.
- Do **not** use an erasable pen or correction fluid.
- Do **not** write on any bar codes.
- If additional space is needed, you should use the lined page at the end of this booklet; the question number or numbers must be clearly shown.
- You should use a calculator where appropriate.
- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

**INFORMATION**

- The total mark for this paper is 75.
- The number of marks for each question or part question is shown in brackets [ ].

This document has **16** pages.

**1** Solve the equation $4|5^x - 1| = 5^x$, giving your answers correct to 3 decimal places. [4]

**2** **(a)** Express $5\sin x - 3\cos x$ in the form $R\sin(x - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{1}{2}\pi$. Give the exact value of $R$ and give $\alpha$ correct to 2 decimal places. [3]

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**(b)** Hence state the greatest and least possible values of $(5\sin x - 3\cos x)^2$. [2]

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**3** The curve with equation $y = xe^{1-2x}$ has one stationary point.

**(a)** Find the coordinates of this point. [4]

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**(b)** Determine whether the stationary point is a maximum or a minimum. [2]

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**4** Using the substitution $u = \sqrt{x}$, find the exact value of

$$\int_3^\infty \frac{1}{(x+1)\sqrt{x}}\,\mathrm{d}x. \qquad [6]$$

**5** **(a)** Show that the equation

$$\cot 2\theta + \cot \theta = 2$$

can be expressed as a quadratic equation in $\tan \theta$. [3]

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**(b)** Hence solve the equation $\cot 2\theta + \cot \theta = 2$, for $0 < \theta < \pi$, giving your answers correct to 3 decimal places. [3]

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**6** When $(a + bx)\sqrt{1 + 4x}$, where $a$ and $b$ are constants, is expanded in ascending powers of $x$, the coefficients of $x$ and $x^2$ are 3 and $-6$ respectively.

Find the values of $a$ and $b$. [6]

**7** **(a)** Given that $y = \ln(\ln x)$, show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x \ln x}.$$ [1]

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The variables $x$ and $t$ satisfy the differential equation

$$x \ln x + t\frac{\mathrm{d}x}{\mathrm{d}t} = 0.$$

It is given that $x = \mathrm{e}$ when $t = 2$.

**(b)** Solve the differential equation obtaining an expression for $x$ in terms of $t$, simplifying your answer. [7]

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**(c)** Hence state what happens to the value of $x$ as $t$ tends to infinity. [1]

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**8** The constant $a$ is such that $\int_1^a \frac{\ln x}{\sqrt{x}}\,\mathrm{d}x = 6$.

**(a)** Show that $a = \exp\left(\frac{1}{\sqrt{a}} + 2\right)$. [5]

[$\exp(x)$ is an alternative notation for $\mathrm{e}^x$.]

**(b)** Verify by calculation that $a$ lies between 9 and 11. [2]

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**(c)** Use an iterative formula based on the equation in part **(a)** to determine $a$ correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

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**9** Two lines $l$ and $m$ have equations $\mathbf{r} = 3\mathbf{i} + 2\mathbf{j} + 5\mathbf{k} + s(4\mathbf{i} - \mathbf{j} + 3\mathbf{k})$ and $\mathbf{r} = \mathbf{i} - \mathbf{j} - 2\mathbf{k} + t(-\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$ respectively.

**(a)** Show that $l$ and $m$ are perpendicular. [2]

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**(b)** Show that $l$ and $m$ intersect and state the position vector of the point of intersection. [5]

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**(c)** Show that the length of the perpendicular from the origin to the line $m$ is $\frac{1}{3}\sqrt{5}$. [4]

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**10** The complex number $1 + 2i$ is denoted by $u$. The polynomial $2x^3 + ax^2 + 4x + b$, where $a$ and $b$ are real constants, is denoted by $p(x)$. It is given that $u$ is a root of the equation $p(x) = 0$.

**(a)** Find the values of $a$ and $b$. [4]

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**(b)** State a second complex root of this equation. [1]

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**(c)** Find the real factors of $\mathrm{p}(x)$. [2]

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**(d)** **(i)** On a sketch of an Argand diagram, shade the region whose points represent complex numbers $z$ satisfying the inequalities $|z - u| \leqslant \sqrt{5}$ and $\arg z \leqslant \frac{1}{4}\pi$. [4]

**(ii)** Find the least value of $\mathrm{Im}\, z$ for points in the shaded region. Give your answer in an exact form. [1]

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## Additional Page

If you use the following lined page to complete the answer(s) to any question(s), the question number(s) must be clearly shown.

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## 5.1 2021 | Oct/Nov | Variant 1 | 9709_w21_ms_31

Cambridge Assessment International Education

# Cambridge International A Level

**MATHEMATICS** **9709/31**

Paper 3 Pure Mathematics 3 **October/November 2021**

MARK SCHEME

Maximum Mark: 75

**Published**

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

Cambridge International will not enter into discussions about these mark schemes.

Cambridge International is publishing the mark schemes for the October/November 2021 series for most Cambridge IGCSE™, Cambridge International A and AS Level components and some Cambridge O Level components.





**[Turn over**

**Generic Marking Principles**

These general marking principles must be applied by all examiners when marking candidate answers. They should be applied alongside the specific content of the mark scheme or generic level descriptors for a question. Each question paper and mark scheme will also comply with these marking principles.

GENERIC MARKING PRINCIPLE 1:
Marks must be awarded in line with:

- the specific content of the mark scheme or the generic level descriptors for the question
- the specific skills defined in the mark scheme or in the generic level descriptors for the question
- the standard of response required by a candidate as exemplified by the standardisation scripts.

GENERIC MARKING PRINCIPLE 2:
Marks awarded are always **whole marks** (not half marks, or other fractions).

GENERIC MARKING PRINCIPLE 3:
Marks must be awarded **positively**:

- marks are awarded for correct/valid answers, as defined in the mark scheme. However, credit is given for valid answers which go beyond the scope of the syllabus and mark scheme, referring to your Team Leader as appropriate
- marks are awarded when candidates clearly demonstrate what they know and can do
- marks are not deducted for errors
- marks are not deducted for omissions
- answers should only be judged on the quality of spelling, punctuation and grammar when these features are specifically assessed by the question as indicated by the mark scheme. The meaning, however, should be unambiguous.

GENERIC MARKING PRINCIPLE 4:
Rules must be applied consistently, e.g. in situations where candidates have not followed instructions or in the application of generic level descriptors.

GENERIC MARKING PRINCIPLE 5:
Marks should be awarded using the full range of marks defined in the mark scheme for the question (however; the use of the full mark range may be limited according to the quality of the candidate responses seen).

GENERIC MARKING PRINCIPLE 6:
Marks awarded are based solely on the requirements as defined in the mark scheme. Marks should not be awarded with grade thresholds or grade descriptors in mind.

| Mathematics Specific Marking Principles | |
|---|---|
| 1 | Unless a particular method has been specified in the question, full marks may be awarded for any correct method. However, if a calculation is required then no marks will be awarded for a scale drawing. |
| 2 | Unless specified in the question, answers may be given as fractions, decimals or in standard form. Ignore superfluous zeros, provided that the degree of accuracy is not affected. |
| 3 | Allow alternative conventions for notation if used consistently throughout the paper, e.g. commas being used as decimal points. |
| 4 | Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored (isw). |
| 5 | Where a candidate has misread a number in the question and used that value consistently throughout, provided that number does not alter the difficulty or the method required, award all marks earned and deduct just 1 mark for the misread. |
| 6 | Recovery within working is allowed, e.g. a notation error in the working where the following line of working makes the candidate's intent clear. |

**Mark Scheme Notes**

The following notes are intended to aid interpretation of mark schemes in general, but individual mark schemes may include marks awarded for specific reasons outside the scope of these notes.

**Types of mark**

**M** Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.

**A** Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).

**B** Mark for a correct result or statement independent of method marks.

**DM** or **DB** When a part of a question has two or more 'method' steps, the M marks are generally independent unless the scheme specifically says otherwise; and similarly, when there are several B marks allocated. The notation DM or DB is used to indicate that a particular M or B mark is dependent on an earlier M or B (asterisked) mark in the scheme. When two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.

**FT** Implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A or B marks are given for correct work only.

- A or B marks are given for correct work only (not for results obtained from incorrect working) unless follow through is allowed (see abbreviation FT above).
- For a numerical answer, allow the A or B mark if the answer is correct to 3 significant figures or would be correct to 3 significant figures if rounded (1 decimal place for angles in degrees).
- The total number of marks available for each question is shown at the bottom of the Marks column.
- Wrong or missing units in an answer should not result in loss of marks unless the guidance indicates otherwise.
- Square brackets [ ] around text or numbers show extra information not needed for the mark to be awarded.

**Abbreviations**

| | |
|---|---|
| AEF/OE | Any Equivalent Form (of answer is equally acceptable) / Or Equivalent |
| AG | Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid) |
| CAO | Correct Answer Only (emphasising that no 'follow through' from a previous error is allowed) |
| CWO | Correct Working Only |
| ISW | Ignore Subsequent Working |
| SOI | Seen Or Implied |
| SC | Special Case (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance) |
| WWW | Without Wrong Working |
| AWRT | Answer Which Rounds To |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 1 | State or imply non-modular equation $4^2\left(5^x-1\right)^2=\left(5^x\right)^2$ or pair of equations $4\left(5^x-1\right)=\pm 5^x$ | **M1** | |
| | Obtain $5^x=\frac{4}{3}$ and $5^x=\frac{4}{5}$ (or $5^{x+1}=4$) | **A1** | |
| | Use correct method for solving an equation of the form $5^x=a$, or $5^{x+1}=b$ where $a>0$, or $b>0$ | **M1** | |
| | Obtain answers $x = 0.179$ and $x = -0.139$ | **A1** | |
| | **Alternative method for question 1** | | |
| | Obtain $5^x=\frac{4}{3}$ by solving an equation | **B1** | |
| | Obtain $5^x=\frac{4}{5}$ (or $5^{x+1}=4$) by solving an equation | **B1** | |
| | Use correct method for solving an equation of the form $5^x=a$, or $5^{x+1}=b$ where $a>0$, or $b>0$ | **M1** | |
| | Obtain answers $x = 0.179$ and $x = -0.139$ | **A1** | |
| | | **4** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 2(a) | State $R = \sqrt{34}$ | **B1** | |
| | Use trig formulae to find $\alpha$ | **M1** | $\tan\alpha = \frac{3}{5}$ or $\sin\alpha = \frac{3}{\sqrt{34}}$ or $\cos\alpha = \frac{5}{\sqrt{34}}$ . |
| | Obtain $\alpha = 0.54$ | **A1** | $30.96°$ scores **M1A0**. |
| | | **3** | |
| 2(b) | State greatest value 34 | **B1 FT** | *Their* $R^2$ . |
| | State least value 0 | **B1** | |
| | | **2** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 3(a) | Use correct product rule | **M1** | |
| | Obtain correct derivative in any form | **A1** | $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^{1-2x} - 2x\mathrm{e}^{1-2x}$ |
| | Equate derivative to zero and solve for $x$ | **M1** | |
| | Obtain $x = \frac{1}{2}$ and $y = \frac{1}{2}$ | **A1** | |
| | | **4** | |
| 3(b) | Use a correct method for determining the nature of a stationary point | **M1** | e.g. $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -2\mathrm{e}^{1-2x} - 2(1-2x)\mathrm{e}^{1-2x}$ |
| | Show that it is a maximum point | **A1** | |
| | | **2** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 4 | State that $\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{2\sqrt{x}}$ or $\mathrm{d}u = \frac{1}{2\sqrt{x}}\,\mathrm{d}x$ | **B1** | |
| | Substitute throughout for $x$ and $\mathrm{d}x$ | **M1** | |
| | Obtain a correct integral with integrand $\frac{2}{u^2+1}$ | **A1** | |
| | Integrate and obtain term of the form $k\tan^{-1}u$ | **M1** | $\left(2\tan^{-1}u\right)$ |
| | Use limits $\sqrt{3}$ and $\infty$ for $u$ or equivalent and evaluate trig. | **A1** | e.g. $2\left(\frac{\pi}{2}-\frac{\pi}{3}\right)$ Must be working in radians. |
| | Obtain answer $\frac{1}{3}\pi$ | **A1** | Or equivalent single term. |
| | | **6** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 5(a) | Use correct trig formulae and express equation in terms of $\tan\theta$ | **M1** | |
| | Obtain a correct equation in $\tan\theta$ in any form | **A1** | e.g. $\frac{1-\tan^2\theta}{2\tan\theta}+\frac{1}{\tan\theta}=2$ |
| | Reduce to $\tan^2\theta+4\tan\theta-3=0$, or 3-term equivalent | **A1** | |
| | | **3** | |
| 5(b) | Solve a 3-term quadratic for tan $\theta$ and calculate $\theta$ | **M1** | $\left(\tan\theta=-2\pm\sqrt{7}\right)$ |
| | Obtain answer, e.g. 0.573 | **A1** | Must be 3 d.p. |
| | Obtain second answer, e.g. 1.783 and no other | **A1** | Ignore answers outside the given interval. Treat answers in degrees as a misread. $(32.9°,102.1°)$ |
| | | **3** | |

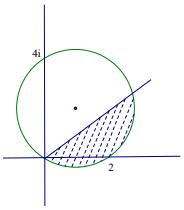
| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 6 | State or imply $1 + 2x$ as first terms of the expansion of $\sqrt{1+4x}$ | **B1** | Allow for correct unsimplified expression. |
| | State or imply $-2x^2$ as third term of the expansion of $\sqrt{1+4x}$ | **B1** | Allow for correct unsimplified expression. |
| | Form an expression for the coefficient of $x$ or coefficient of $x^2$ in the expansion of $(a+bx)\sqrt{1+4x}$ and equate to given coefficient | **M1** | All relevant terms considered. |
| | Obtain $2a + b = 3$, or equivalent | **A1** | One correct equation. |
| | Obtain $-2a + 2b = -6$ or equivalent | **A1** | Second correct equation. |
| | Obtain answer $a = 2$ and $b = -1$ | **A1** | |
| | | **6** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 7(a) | Show sufficient working to justify the given answer | **B1** | |
| | | **1** | |
| 7(b) | Correct separation of variables | **B1** | e.g. $-\int\frac{1}{t}\mathrm{d}t=\int\frac{1}{x\ln x}\mathrm{d}x$ |
| | Obtain term $\ln(\ln x)$ | **B1** | |
| | Obtain term $-\ln t$ | **B1** | |
| | Evaluate a constant or use $x$ = e and $t$ = 2 as limits in an expression involving $\ln(\ln x)$ | **M1** | |
| | Obtain correct solution in any form, e.g. $\ln(\ln x)=-\ln t+\ln 2$ | **A1** | |
| | Use log laws to enable removal of logarithms | **M1** | |
| | Obtain answer $x=\mathrm{e}^{\frac{2}{t}}$, or simplified equivalent | **A1** | |
| | | **7** | |
| 7(c) | State that $x$ tends to 1 coming from $x=\mathrm{e}^{\frac{k}{t}}$ | **B1** | |
| | | **1** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 8(a) | Commence integration and reach $a\sqrt{x}\ln x + b\int \sqrt{x}\ .\ \frac{1}{x}\ \mathrm{d}x$, or equivalent | ***M1** | |
| | Obtain $2\sqrt{x}\ln x - \int 2\sqrt{x}\ .\ \frac{1}{x}\ \mathrm{d}x$, or equivalent | **A1** | |
| | Obtain integral $2\sqrt{x}\ln x - 4\sqrt{x}$ , or equivalent | **A1** | |
| | Substitute limits and equate result to 6 | **DM1** | |
| | Rearrange and obtain $a = \exp\left(\frac{1}{\sqrt{a}} + 2\right)$ | **A1** | Obtain **given answer** from full and correct working. |
| | | **5** | |
| 8(b) | Calculate the values of a relevant expression or pair of expressions at $a = 9$ and $a = 11$ | **M1** | e.g. $\begin{cases} 9 < 10.31 \\ 11 > 9.99 \end{cases}$ or $1.31 > 0, -1.01 < 0$ |
| | Complete the argument correctly with correct values | **A1** | |
| | | **2** | |
| 8(c) | Use the iterative process $a_{n+1} = \exp(\ \frac{1}{\sqrt{a_n}}\ + 2)$ correctly at least once | **M1** | |
| | Obtain answer 10.12 | **A1** | |
| | Show sufficient iterations to 4dp to justify 10.12 to 2dp, or show there is a sign change in the interval (10.115, 10.125) | **A1** | e.g. $10, 10.1374, 10.1156, 10.1190, ....$<br>$9, 10.3123, 10.0886, 10.1233, 10.1178, ...$<br>$11, 9.9893, 10.1391, 10.1153, 10.1191, ....$ |
| | | **3** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 9(a) | Use correct method to evaluate the scalar product of relevant vectors | **M1** | $(-4-2+6)$ |
| | Obtain answer zero and deduce the given statement | **A1** | Need a conclusion or a statement in advance that the scalar product will be zero. |
| | | **2** | |
| 9(b) | Express general point of $l$ or $m$ in component form, e.g. $(3+4s, 2-s, 5+3s)$ or $(1-t, -1+2t, -2+2t)$ | **B1** | |
| | Equate at least two pairs of components and solve for $s$ or for $t$ | **M1** | |
| | Obtain correct answer $s = -1$ and $t = 2$ | **A1** | |
| | Verify that all three equations are satisfied | **A1** | |
| | State position vector of the intersection $-\mathbf{i}+3\mathbf{j}+2\mathbf{k}$, or equivalent | **A1** | Can come from 1 correct value and no contradictory statement. |
| | | **5** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 9(c) | Taking a general point $P$ on $m$, form an equation in $t$ by ***either*** equating a relevant scalar product to zero, ***or*** equating the derivative of $\lvert\overrightarrow{OP}\rvert$ to zero, ***or*** taking a specific point $Q$ on $m$, e.g. (1, – 1, – 2), using Pythagoras in triangle $OPQ$ | ***M1** | e.g. $\begin{pmatrix}1-t\\-1+2t\\-2+2t\end{pmatrix}.\begin{pmatrix}-1\\2\\2\end{pmatrix}=0$ |
| | Obtain $t=\frac{7}{9}$ | **A1** | |
| | Carry out correct method to find $OP$ | **DM1** | |
| | Obtain $\frac{\sqrt{5}}{3}$ | **A1** | Obtain the **given answer** from full and correct working. |
| | **Alternative method for question 9(c)** | | |
| | Take a specific point $Q$ on $m$, e.g. (– 1, 3, 2) and use a scalar product to find $QN$, the projection of $OQ$ on $m$ | ***M1** | |
| | Obtain $QN=\frac{11}{3}$, or equivalent | **A1** | |
| | Use Pythagoras to obtain $ON$ | **DM1** | |
| | Obtain the given answer correctly | **A1** | |
| | | **4** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 10(a) | Substitute 1 + 2i in the polynomial and attempt expansions of $x^2$ and $x^3$ | **M1** | $u^2 = -3 + 4\text{i},\ u^3 = -11 - 2\text{i}$<br>Full substitution but need not simplify. |
| | Equate real and/or imaginary parts to zero | **M1** | $-18 - 3a + b = 0,\ 4 + 4a = 0$ |
| | Obtain $a = -1$ | **A1** | |
| | Obtain $b = 15$ | **A1** | |
| | | **4** | |
| 10(b) | State second root 1 – 2i | **B1** | |
| | | **1** | |
| 10(c) | State the quadratic factor $x^2 - 2x + 5$ | **B1** | |
| | State the linear factor $2x + 3$ | **B1** | |
| | | **2** | |
| 10(d)(i) | Show a circle with centre 1 + 2i | **B1** |  |
| | Show circle passing through the origin | **B1** | |
| | Show the half line $y = x$ in the first quadrant (accept chord of circle) | **B1** | |
| | Shade the correct region on a correct diagram | **B1** | |
| | | **4** | |
| 10(d)(ii) | State answer $2 - \sqrt{5}$ | **B1** | |
| | | **1** | |

# 6 2021 | Oct/Nov | Variant 2 | 9709_w21_qp_32

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Cambridge Assessment International Education

# Cambridge International AS & A Level

CANDIDATE NAME

CENTRE NUMBER

CANDIDATE NUMBER

*4325784148*

**MATHEMATICS** **9709/32**

Paper 3 Pure Mathematics 3 **October/November 2021**

**1 hour 50 minutes**

You must answer on the question paper.

You will need: List of formulae (MF19)

**INSTRUCTIONS**

- Answer **all** questions.
- Use a black or dark blue pen. You may use an HB pencil for any diagrams or graphs.
- Write your name, centre number and candidate number in the boxes at the top of the page.
- Write your answer to each question in the space provided.
- Do **not** use an erasable pen or correction fluid.
- Do **not** write on any bar codes.
- If additional space is needed, you should use the lined page at the end of this booklet; the question number or numbers must be clearly shown.
- You should use a calculator where appropriate.
- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

**INFORMATION**

- The total mark for this paper is 75.
- The number of marks for each question or part question is shown in brackets [ ].

This document has **20** pages. Any blank pages are indicated.

**BLANK PAGE**

**1** Find the value of $x$ for which $3(2^{1-x}) = 7^x$. Give your answer in the form $\frac{\ln a}{\ln b}$, where $a$ and $b$ are integers. [4]

**2** Solve the inequality $|3x - a| > 2|x + 2a|$, where $a$ is a positive constant. [4]

**3** **(a)** Given the complex numbers $u = a + \mathrm{i}b$ and $w = c + \mathrm{i}d$, where $a$, $b$, $c$ and $d$ are real, prove that $(u + w)^* = u^* + w^*$. [2]

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**(b)** Solve the equation $(z + 2 + \mathrm{i})^* + (2 + \mathrm{i})z = 0$, giving your answer in the form $x + \mathrm{i}y$ where $x$ and $y$ are real. [4]

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**4** Express $\dfrac{4x^2 - 13x + 13}{(2x - 1)(x - 3)}$ in partial fractions. [5]

**5** **(a)** On a sketch of an Argand diagram, shade the region whose points represent complex numbers $z$ satisfying the inequalities $|z - 3 - 2i| \leqslant 1$ and $\text{Im}\, z \geqslant 2$. [4]

**(b)** Find the greatest value of $\arg z$ for points in the shaded region, giving your answer in degrees. [3]

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**6** **(a)** Using the expansions of $\sin(3x + 2x)$ and $\sin(3x - 2x)$, show that

$$\tfrac{1}{2}(\sin 5x + \sin x) \equiv \sin 3x \cos 2x. \qquad [3]$$

**(b)** Hence show that $\int_0^{\frac{1}{4}\pi} \sin 3x \cos 2x \, \mathrm{d}x = \frac{1}{5}(3 - \sqrt{2})$. [3]

**7** The variables $x$ and $y$ satisfy the differential equation

$$e^{2x}\frac{dy}{dx} = 4xy^2,$$

and it is given that $y = 1$ when $x = 0$.

Solve the differential equation, obtaining an expression for $y$ in terms of $x$. [7]

**8** **(a)** By first expanding $\left(\cos^2\theta + \sin^2\theta\right)^2$, show that

$$\cos^4\theta + \sin^4\theta \equiv 1 - \tfrac{1}{2}\sin^2 2\theta.$$ [3]

**(b)** Hence solve the equation

$$\cos^4\theta + \sin^4\theta = \tfrac{5}{9},$$

for $0° < \theta < 180°$. [4]

**9** The equation of a curve is $ye^{2x} - y^2e^x = 2$.

**(a)** Show that $\frac{dy}{dx} = \frac{2ye^x - y^2}{2y - e^x}$. [4]

**(b)** Find the exact coordinates of the point on the curve where the tangent is parallel to the $y$-axis. [4]

**10** With respect to the origin $O$, the position vectors of the points $A$ and $B$ are given by $\overrightarrow{OA} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$.

**(a)** Find a vector equation for the line $l$ through $A$ and $B$. [3]

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**(b)** The point $C$ lies on $l$ and is such that $\overrightarrow{AC} = 3\overrightarrow{AB}$.

Find the position vector of $C$. [2]

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**(c)** Find the possible position vectors of the point $P$ on $l$ such that $OP = \sqrt{14}$. [5]

**11** The equation of a curve is $y = \sqrt{\tan x}$, for $0 \leqslant x < \frac{1}{2}\pi$.

**(a)** Express $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of $\tan x$, and verify that $\frac{\mathrm{d}y}{\mathrm{d}x} = 1$ when $x = \frac{1}{4}\pi$. [4]

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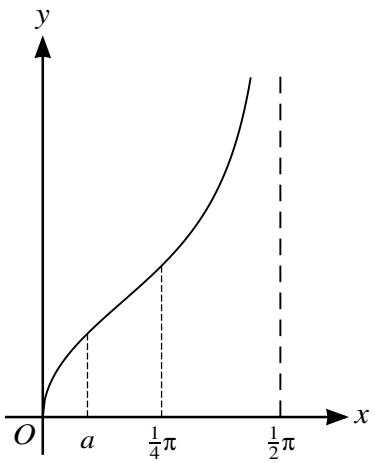
The value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ is also 1 at another point on the curve where $x = a$, as shown in the diagram.




**(b)** Show that $t^3 + t^2 + 3t - 1 = 0$, where $t = \tan a$. [4]

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**(c)** Use the iterative formula

$$a_{n+1} = \tan^{-1}\left(\tfrac{1}{3}(1 - \tan^2 a_n - \tan^3 a_n)\right)$$

to determine $a$ correct to 2 decimal places, giving the result of each iteration to 4 decimal places. [3]

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## Additional Page

If you use the following lined page to complete the answer(s) to any question(s), the question number(s) must be clearly shown.

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### 6.1 2021 | Oct/Nov | Variant 2 | 9709_w21_ms_32

Cambridge Assessment International Education

# Cambridge International A Level

**MATHEMATICS** **9709/32**

Paper 3 Pure Mathematics 3 **October/November 2021**

MARK SCHEME

Maximum Mark: 75

**Published**

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

Cambridge International will not enter into discussions about these mark schemes.

Cambridge International is publishing the mark schemes for the October/November 2021 series for most Cambridge IGCSE™, Cambridge International A and AS Level components and some Cambridge O Level components.

**Generic Marking Principles**

These general marking principles must be applied by all examiners when marking candidate answers. They should be applied alongside the specific content of the mark scheme or generic level descriptors for a question. Each question paper and mark scheme will also comply with these marking principles.

GENERIC MARKING PRINCIPLE 1:
Marks must be awarded in line with:

- the specific content of the mark scheme or the generic level descriptors for the question
- the specific skills defined in the mark scheme or in the generic level descriptors for the question
- the standard of response required by a candidate as exemplified by the standardisation scripts.

GENERIC MARKING PRINCIPLE 2:
Marks awarded are always **whole marks** (not half marks, or other fractions).

GENERIC MARKING PRINCIPLE 3:
Marks must be awarded **positively**:

- marks are awarded for correct/valid answers, as defined in the mark scheme. However, credit is given for valid answers which go beyond the scope of the syllabus and mark scheme, referring to your Team Leader as appropriate
- marks are awarded when candidates clearly demonstrate what they know and can do
- marks are not deducted for errors
- marks are not deducted for omissions
- answers should only be judged on the quality of spelling, punctuation and grammar when these features are specifically assessed by the question as indicated by the mark scheme. The meaning, however, should be unambiguous.

GENERIC MARKING PRINCIPLE 4:
Rules must be applied consistently, e.g. in situations where candidates have not followed instructions or in the application of generic level descriptors.

GENERIC MARKING PRINCIPLE 5:
Marks should be awarded using the full range of marks defined in the mark scheme for the question (however; the use of the full mark range may be limited according to the quality of the candidate responses seen).

GENERIC MARKING PRINCIPLE 6:
Marks awarded are based solely on the requirements as defined in the mark scheme. Marks should not be awarded with grade thresholds or grade descriptors in mind.

| Mathematics Specific Marking Principles | |
|---|---|
| 1 | Unless a particular method has been specified in the question, full marks may be awarded for any correct method. However, if a calculation is required then no marks will be awarded for a scale drawing. |
| 2 | Unless specified in the question, answers may be given as fractions, decimals or in standard form. Ignore superfluous zeros, provided that the degree of accuracy is not affected. |
| 3 | Allow alternative conventions for notation if used consistently throughout the paper, e.g. commas being used as decimal points. |
| 4 | Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored (isw). |
| 5 | Where a candidate has misread a number in the question and used that value consistently throughout, provided that number does not alter the difficulty or the method required, award all marks earned and deduct just 1 mark for the misread. |
| 6 | Recovery within working is allowed, e.g. a notation error in the working where the following line of working makes the candidate's intent clear. |

**Mark Scheme Notes**

The following notes are intended to aid interpretation of mark schemes in general, but individual mark schemes may include marks awarded for specific reasons outside the scope of these notes.

**Types of mark**

**M** Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.

**A** Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).

**B** Mark for a correct result or statement independent of method marks.

**DM** or **DB** When a part of a question has two or more 'method' steps, the M marks are generally independent unless the scheme specifically says otherwise; and similarly, when there are several B marks allocated. The notation DM or DB is used to indicate that a particular M or B mark is dependent on an earlier M or B (asterisked) mark in the scheme. When two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.

**FT** Implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A or B marks are given for correct work only.

- A or B marks are given for correct work only (not for results obtained from incorrect working) unless follow through is allowed (see abbreviation FT above).
- For a numerical answer, allow the A or B mark if the answer is correct to 3 significant figures or would be correct to 3 significant figures if rounded (1 decimal place for angles in degrees).
- The total number of marks available for each question is shown at the bottom of the Marks column.
- Wrong or missing units in an answer should not result in loss of marks unless the guidance indicates otherwise.
- Square brackets [ ] around text or numbers show extra information not needed for the mark to be awarded.

**Abbreviations**

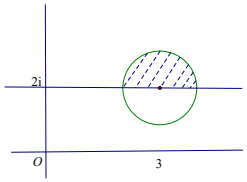
| | |
|---|---|
| AEF/OE | Any Equivalent Form (of answer is equally acceptable) / Or Equivalent |
| AG | Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid) |
| CAO | Correct Answer Only (emphasising that no 'follow through' from a previous error is allowed) |
| CWO | Correct Working Only |
| ISW | Ignore Subsequent Working |
| SOI | Seen Or Implied |
| SC | Special Case (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance) |
| WWW | Without Wrong Working |
| AWRT | Answer Which Rounds To |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 1 | Use law of the logarithm of a product, a quotient or power | ***M1** | e.g. $\ln\left(7^x\right) = x\ln 7$ |
| | Obtain a correct linear equation in any form | **A1** | e.g. $\ln 3 + (1-x)\ln 2 = x\ln 7$ |
| | Solve a linear equation for $x$ | **DM1** | |
| | Obtain answer $x = \dfrac{\ln 6}{\ln 14}$ | **A1** | Maximum 3 out of 4 available if final answer not in required form e.g. 0.67…<br>ISW once correct answer seen. |
| | **Alternative method for Question 1** | | |
| | $2^{1-x} = 2\times 2^{-x}$ | ***M1** | OE |
| | $6 = 2^x 7^x \left[= 14^x\right]$ | **A1** | |
| | Use law of the logarithm of a power to solve for $x$ | **DM1** | Must be a linear power. Allow $x = \ln_{14}(6)$. |
| | Obtain answer $x = \dfrac{\ln 6}{\ln 14}$ | **A1** | ISW once correct answer seen. |
| | | **4** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 2 | State or imply non-modular inequality $(3x-a)^2 > 2^2(x+2a)^2$, or corresponding quadratic equation, or pair of linear equations or linear inequalities | **B1** | Need $2^2$ seen or implied. |
| | Make reasonable attempt to solve a 3-term quadratic, or solve two linear equations for $x$ in terms of $a$ | **M1** | $\left(5x^2-22ax-15a^2=0\right)$ |
| | Obtain critical values $x = 5a$ and $x = -\frac{3}{5}a$ and no others | **A1** | OE<br>Accept incorrect inequalities with correct critical values.<br>Must state 2 values i.e. $\frac{a\pm b}{c}$ is not sufficient. |
| | State final answer $x > 5a$, $x < -\frac{3}{5}a$ | **A1** | Do not condone $\geqslant$ for > or $\leqslant$ for < in the final answer.<br>$5a < x < -\frac{3}{5}a$ is **A0**, 'and' is **A0**. |
| | **Alternative method for Question 2** | | |
| | Obtain critical value $x = 5a$ from a graphical method, or by solving a linear equation or linear inequality | **B1** | |
| | Obtain critical value $x = -\frac{3}{5}a$ similarly | **B2** | Maximum 2 marks if more than 2 critical values. |
| | State final answer $x > 5a$ , $x < -\frac{3}{5}a$ | **B1** | Do not condone $\geq$ for > or $\leq$ for < in the final answer.<br>$5a < x < -\frac{3}{5}a$ is **B0**, 'and' is **B0**. |
| | | **4** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 3(a) | Substitute for $u$ and $w$ and state correct conjugate of one side | **B1** | |
| | Express the other side without conjugates and confirm $(u+w)^* = u^* + w^*$ | **B1** | **Given answer.** Needs explicit reference to conjugate of both sides. |
| | | **2** | |
| 3(b) | Substitute and remove conjugates to obtain a correct equation in $x$ and $y$ | **B1** | e.g. $x + 2 - (y+1)\mathrm{i} + (2+\mathrm{i})(x+\mathrm{i}y) = 0$ |
| | Use $\mathrm{i}^2 = -1$ and equate real and imaginary parts to zero | **M1** | |
| | Obtain two correct equations in $x$ and $y$ | **A1** | e.g. $3x - y + 2 = 0$ and $x + y - 1 = 0$. Allow $x\mathrm{i} + y\mathrm{i} - \mathrm{i} = 0$. |
| | Solve and obtain answer $z = -\frac{1}{4} + \frac{5}{4}\mathrm{i}$ | **A1** | Allow for real and imaginary parts stated separately. |
| | | **4** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 4 | State or imply the form $A + \frac{B}{2x-1} + \frac{C}{x-3}$ | **B1** | $\frac{Dx+E}{2x-1} + \frac{F}{x-3}$ and $\frac{P}{2x-1} + \frac{Qx+R}{x-3}$ are also valid. |
| | Use a correct method for finding a constant | **M1** | |
| | Obtain one of $A = 2$, $B = -3$ and $C = 2$ | **A1** | Allow maximum **M1A1** for one or more 'correct' values after **B0**. |
| | Obtain a second value | **A1** | |
| | Obtain the third value | **A1** | |
| | **Alternative method for Question 4** | | |
| | Divide numerator by denominator | **M1** | |
| | Obtain $2\left[+\frac{Px+Q}{(2x-1)(x-3)}\right]$ | **A1** | $\left[2+\frac{x+7}{(2x-1)(x-3)}\right]$ |
| | State or imply the form $\frac{D}{2x-1} + \frac{E}{x-3}$ | **B1** | |
| | Obtain one of $D = -3$ and $E = 2$ | **A1** | |
| | Obtain a second value | **A1** | |
| | | **5** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 5(a) | Show circle with centre 3 + 2i | **B1** |  |
| | Show circle with radius 1. Must match *their* scales: if scales not identical should have an ellipse. | **B1** | |
| | Show line $y = 2$ in at least the diameter of a circle in the first quadrant | **B1** | |
| | Shade the correct region in a correct diagram | **B1** | |
| | | **4** | |
| 5(b) | Identify the correct point | **B1** | |
| | Carry out a correct method for finding the argument | **M1** | e.g. $\arg x = \tan^{-1}\frac{2}{3} + \sin^{-1}\frac{1}{\sqrt{13}}$<br>Exact working required. |
| | Obtain answer 49.8° | **A1** | Or better. 0.869 radians scores **B1M1A0**. |
| | | **3** | **Special Case 1: B1M0** for 45° if they have shaded the wrong half of the circle.<br>**Special Case 2:** 3 out of 3 available if they identify the correct point on the correct circle and it is consistent with *their* shading. |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 6(a) | State correct expansion of $\sin(3x+2x)$ or $\sin(3x-2x)$ | **B1** | |
| | Substitute expansions in $\frac{1}{2}(\sin 5x+\sin x)$, or equivalent | **M1** | |
| | Simplify and obtain $\frac{1}{2}(\sin 5x+\sin x)=\sin 3x\cos 2x$ | **A1** | Obtain the **given identity** correctly. |
| | | **3** | |
| 6(b) | Obtain integral $-\frac{1}{10}\cos 5x-\frac{1}{2}\cos x$, or equivalent | **B1** | |
| | Substitute limits correctly in an expression of the form $p\cos 5x+q\cos x$ | **M1** | Correct limits and subtracted the right way around.<br>Do not need values of trig functions for M1.<br>Maximum one slip. |
| | Obtain $\frac{1}{5}\left(3-\sqrt{2}\right)$ | **A1** | Substitute values and obtain the **given answer** following full, correct and exact working. |
| | | **3** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 7 | Separate variables correctly | **B1** | $\int \frac{1}{y^2}\mathrm{d}y = \int 4x\mathrm{e}^{-2x}\mathrm{d}x$ |
| | $\int \frac{1}{y^2}\mathrm{d}y = -\frac{1}{y}$ | **B1** | OE |
| | Commence the other integration and reach $ax\mathrm{e}^{-2x} + b\int \mathrm{e}^{-2x}\,\mathrm{d}x$ | **M1** | |
| | Obtain $-2x\mathrm{e}^{-2x} + 2\int \mathrm{e}^{-2x}\,\mathrm{d}x$ or $-\frac{1}{2}x\mathrm{e}^{-2x} + \frac{1}{2}\int \mathrm{e}^{-2x}\,\mathrm{d}x$ | **A1** | SOI (might have taken out factor of 4) |
| | Complete integration and obtain $-2x\mathrm{e}^{-2x} - \mathrm{e}^{-2x}$ | **A1** | |
| | Evaluate a constant or use $x = 0$ and $y = 1$ as limits in a solution containing terms of the form $\frac{p}{y}, qx\mathrm{e}^{-2x}, r\mathrm{e}^{-2x}$, or equivalent. | **M1** | |
| | Obtain $y = \frac{\mathrm{e}^{2x}}{2x+1}$, or equivalent expression for $y$ | **A1** | ISW |
| | | **7** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 8(a) | Expand the square and equate to 1 | **B1** | |
| | Use correct double angle formula | **M1** | Need to see $\frac{4}{2}$ or $\sin 2\theta = 2\sin\theta\cos\theta$ stated. |
| | Obtain $\cos^4\theta + \sin^4\theta = 1 - \frac{1}{2}\sin^2 2\theta$ | **A1** | Obtain the **given result** correctly. |
| | | **3** | |
| 8(b) | Use the identity and carry out a method for finding a root | **M1** | $\left(1 - \frac{1}{2}\sin^2 2\theta = \frac{5}{9}\right)$ |
| | Obtain answer $35.3°$ | **A1** | Must be correct if overspecified: 35.264... |
| | Obtain a second answer, e.g. $54.7°$ | **A1 FT** | $[\text{e.g } 90° - \textit{their } 35.3°]$<br>Do not FT if mixing degrees and radians. |
| | Obtain the remaining answers, e.g. $144.7°$ and $125.3°$ and no others in the given interval | **A1 FT** | $[\text{e.g.} 180° - .. \text{ and } 180° - ..]$<br>Ignore answers outside the given interval.<br>Treat answers in radians as a misread.<br>$(0.615, 0.955, 2.19, 2.53)$<br>Do not FT if mixing degrees and radians. |
| | | **4** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 9(a) | State correct derivative of $ye^{2x}$ with respect to $x$ | **B1** | $2ye^{2x}+e^{2x}\frac{dy}{dx}$ |
| | State correct derivative of $y^2e^x$ with respect to $x$ | **B1** | $2ye^x\frac{dy}{dx}+y^2e^x$ |
| | Equate attempted derivative of the LHS to zero and solve for $\frac{dy}{dx}$ | **M1** | |
| | Obtain $\frac{dy}{dx}=\frac{2ye^x-y^2}{2y-e^x}$ | **A1** | Obtain the **given answer** correctly.<br>Condone multiplication by $\frac{-1}{-1}$ and cancelling of $e^x$ without comment. |
| | **Alternative method for Question 9(a)** | | |
| | Rearrange as $y=\frac{2}{e^{2x}-ye^x} \Rightarrow \frac{d}{dx}\left(e^{2x}-ye^x\right)=2e^{2x}-ye^x-e^x\frac{dy}{dx}$ | **B1** | Other rearrangements are possible e.g.<br>$y=2e^{-2x}+y^2e^{-x}$ $\frac{d}{dx}\left(y^2e^{-x}\right)=2ye^{-x}\frac{dy}{dx}-y^2e^{-x}$ |
| | $\frac{dy}{dx}=-\frac{2}{\left(e^{2x}-ye^x\right)^2}\times\left(2e^{2x}-ye^x-e^x\frac{dy}{dx}\right)$ | **B1** | $\Rightarrow \frac{dy}{dx}=-4e^{-x}+2ye^{-x}\frac{dy}{dx}-y^2e^{-x}$ |
| | Solve for $\frac{dy}{dx}$ | **M1** | |
| | Obtain $\frac{dy}{dx}=\frac{2ye^x-y^2}{2y-e^x}$ | **A1** | Obtain the **given answer** correctly. |
| | | **4** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 9(b) | Equate denominator to zero and substitute for $y$ or for $e^x$ in the equation of the curve | ***M1** | |
| | Obtain equation of the form $ae^{3x} = b$ or $cy^3 = d$ | **DM1** | $\left(e^{3x} = 8, \quad y^3 = 1\right)$ SOI |
| | Obtain $x = \ln 2$ | **A1** | Accept $\frac{1}{3}\ln 8$ ISW |
| | Obtain $y = 1$ | **A1** | |
| | | **4** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 10(a) | Obtain direction vector $-\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ , or equivalent | **B1** | Accept answers as column vectors throughout. |
| | Use a correct method to form a vector equation | **M1** | |
| | State answer $\mathbf{r} = \mathbf{i} + 2\mathbf{j} - \mathbf{k} + \lambda(-\mathbf{i} + \mathbf{j} + 2\mathbf{k})$, or equivalent correct form | **A1** | e.g. $\mathbf{r} = \begin{pmatrix}0\\3\\1\end{pmatrix} + \mu\begin{pmatrix}1\\-1\\-2\end{pmatrix}$ Allow $\begin{pmatrix}x\\y\\z\end{pmatrix}$ for $\mathbf{r}$. |
| | | **3** | |
| 10(b) | Use a correct method to find the position vector of $C$ | **M1** | e.g. $\mathbf{OC} = \mathbf{OA} + \mathbf{AC} = \begin{pmatrix}1-3\\2+3\\-1+6\end{pmatrix}$ |
| | Obtain answer $-2\mathbf{i} + 5\mathbf{j} + 5\mathbf{k}$, or equivalent | **A1** | Accept as coordinates. |
| | | **2** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 10(c) | State $\overrightarrow{OP}$ in component form | **B1 FT** | |
| | Form an equation in $\lambda$ by equating the modulus of $OP$ to $\sqrt{14}$, or equivalent | **M1** | |
| | Simplify and obtain $3\lambda^2 - \lambda - 4 = 0$, or equivalent | **A1** | $3\lambda^2 + \lambda - 4 = 0$ if using $\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ in **(a)**.<br>$3\mu^2 + 5\mu - 2 = 0$ if using $-\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ in **(a)** and $OB$. |
| | Solve a 3-term quadratic and find a position vector | **M1** | $\left(\lambda = -1, \frac{4}{3}\right.$ or $\lambda = 1, -\frac{4}{3}$ or $\mu = \frac{1}{3}, -2$ or $\left.\mu = -\frac{1}{3}, 2\right)$ |
| | Obtain answers $2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ and $-\frac{1}{3}\mathbf{i} + \frac{10}{3}\mathbf{j} + \frac{5}{3}\mathbf{k}$, or equivalent | **A1** | Accept as coordinates. |
| | | **5** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 11(a) | Use chain rule | **M1** | Allow if not starting with the correct index. |
| | Obtain correct derivative in any form | **A1** | e.g. $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sec^2 x}{2\sqrt{\tan x}}$ |
| | Use correct Pythagoras to obtain correct derivative in terms of $\tan x$ | **A1** | e.g. $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1 + \tan^2 x}{2\sqrt{\tan x}}$ |
| | Use a correct derivative to obtain $\frac{\mathrm{d}y}{\mathrm{d}x} = 1$ when $x = \frac{1}{4}\pi$ | **B1** | Confirm the **given statement** from correct work.<br>Should see at least $\frac{2}{2} = 1$. |
| | | **4** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 11(b) | Equate answer to part (**a**) to 1 and obtain a quartic equation in $t$ or $\tan x$ | ***M1** | At least as far as $\left(1+\tan^2 x\right)^2 = 4\tan x$. |
| | Obtain correct answer, i.e. $t^4 + 2t^2 - 4t + 1 = 0$ | **A1** | Or equivalent horizontal form. |
| | Commence division by $t - 1$ | **DM1** | As far as $t^3 + t^2 + \ldots$by long division or inspection.<br>Allow verification by multiplying given answer by $t - 1$. |
| | Obtain the given answer | **A1** | |
| | | **4** | |
| 11(c) | Use the iterative process correctly with the given formula at least once | **M1** | Obtain one value and use that to obtain the next.<br>Must be working in radians. |
| | Obtain final answer $a = 0.29$ | **A1** | |
| | Show sufficient iterations to 4 d.p. to justify 0.29 to 2 d.p., or show there is a sign change in (0.285, 0.295) | **A1** | e.g. 0.3, 0.2854, 0.2894, 0.2883, .....<br>0.4, 0.2436, 0.2984, 0.2841, 0.2883, 0.2871, ...<br>0.5, 0.1776, 0.3103, 0.2805, 0.2893, 0.2868, ... |
| | | **3** | |

# 7 2021 | Oct/Nov | Variant 3 | 9709_w21_qp_33

**Topic & Sub-topic Index of Questions**

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|---|---|---|---|---|---|
| 1 | | Pure Mathematics 3 | Algebra | 229 | 254 |
| 2 | a | Pure Mathematics 3 | Algebra | 230 | 254 |
| 2 | b | Pure Mathematics 3 | Algebra | 230 | 255 |
| 3 | | Pure Mathematics 3 | Logarithmic and exponential functions | 231 | 256 |
| 4 | | Pure Mathematics 3 | Integration | 232 | 256 |
| 5 | | Pure Mathematics 3 | Trigonometry | 233 | 257 |
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| 7 | a | Pure Mathematics 3 | Differentiation | 235 | 258 |
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| 10 | a | Pure Mathematics 3 | Differential equations | 241 | 261 |
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Cambridge Assessment International Education

# Cambridge International AS & A Level

CANDIDATE NAME

CENTRE NUMBER

CANDIDATE NUMBER

*1472641323*

**MATHEMATICS** **9709/33**

Paper 3 Pure Mathematics 3 **October/November 2021**

**1 hour 50 minutes**

You must answer on the question paper.

You will need: List of formulae (MF19)

**INSTRUCTIONS**

- Answer **all** questions.
- Use a black or dark blue pen. You may use an HB pencil for any diagrams or graphs.
- Write your name, centre number and candidate number in the boxes at the top of the page.
- Write your answer to each question in the space provided.
- Do **not** use an erasable pen or correction fluid.
- Do **not** write on any bar codes.
- If additional space is needed, you should use the lined page at the end of this booklet; the question number or numbers must be clearly shown.
- You should use a calculator where appropriate.
- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

**INFORMATION**

- The total mark for this paper is 75.
- The number of marks for each question or part question is shown in brackets [ ].

This document has **20** pages. Any blank pages are indicated.

**1** Find the quotient and remainder when $2x^4 + 1$ is divided by $x^2 - x + 2$. [3]

**2** **(a)** Sketch the graph of $y = |2x - 3|$. [1]

**(b)** Solve the inequality $|2x - 3| < 3x + 2$. [3]

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**3** Solve the equation $4^{x-2} = 4^x - 4^2$, giving your answer correct to 3 decimal places. [4]

**4** Find the exact value of $\int_{\frac{1}{3}\pi}^{\pi} x \sin \frac{1}{2}x \, \mathrm{d}x$. [5]

**5** Solve the equation $\sin\theta = 3\cos 2\theta + 2$, for $0° \leqslant \theta \leqslant 360°$. [5]

**6** **(a)** By first expanding $\cos(x - 60°)$, show that the expression

$$2\cos(x - 60°) + \cos x$$

can be written in the form $R\cos(x - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. Give the exact value of $R$ and the value of $\alpha$ correct to 2 decimal places. [5]

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**(b)** Hence find the value of $x$ in the interval $0° < x < 360°$ for which $2\cos(x - 60°) + \cos x$ takes its least possible value. [2]

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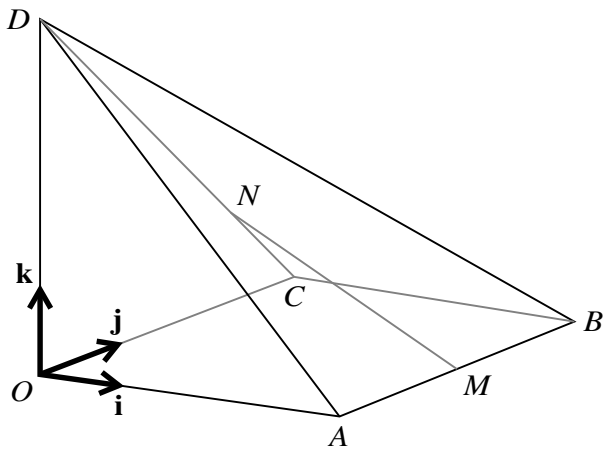
**7** The equation of a curve is $\ln(x + y) = x - 2y$.

**(a)** Show that $\frac{dy}{dx} = \frac{x + y - 1}{2(x + y) + 1}$. [4]

**(b)** Find the coordinates of the point on the curve where the tangent is parallel to the $x$-axis. [3]

**8**




In the diagram, $OABCD$ is a pyramid with vertex $D$. The horizontal base $OABC$ is a square of side 4 units. The edge $OD$ is vertical and $OD = 4$ units. The unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are parallel to $OA$, $OC$ and $OD$ respectively.

The midpoint of $AB$ is $M$ and the point $N$ on $CD$ is such that $DN = 3NC$.

**(a)** Find a vector equation for the line through $M$ and $N$. [5]

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**(b)** Show that the length of the perpendicular from $O$ to $MN$ is $\frac{1}{3}\sqrt{82}$. [4]

**9** Let $\mathrm{f}(x) = \dfrac{1}{(9-x)\sqrt{x}}$.

**(a)** Find the $x$-coordinate of the stationary point of the curve with equation $y = \mathrm{f}(x)$. [4]

**(b)** Using the substitution $u = \sqrt{x}$, show that $\int_0^4 \mathrm{f}(x)\,\mathrm{d}x = \frac{1}{3}\ln 5$. [6]

**10** A large plantation of area 20 $\text{km}^2$ is becoming infected with a plant disease. At time $t$ years the area infected is $x\,\text{km}^2$ and the rate of increase of $x$ is proportional to the ratio of the area infected to the area not yet infected.

When $t = 0$, $x = 1$ and $\dfrac{\mathrm{d}x}{\mathrm{d}t} = 1$.

**(a)** Show that $x$ and $t$ satisfy the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{19x}{20 - x}.$$ [2]

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**(b)** Solve the differential equation and show that when $t = 1$ the value of $x$ satisfies the equation $x = \mathrm{e}^{0.9+0.05x}$. [5]

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**(c)** Use an iterative formula based on the equation in part **(b)**, with an initial value of 2, to determine $x$ correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

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**(d)** Calculate the value of $t$ at which the entire plantation becomes infected. [1]

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**11** The complex number $-\sqrt{3} + \mathrm{i}$ is denoted by $u$.

**(a)** Express $u$ in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$, giving the exact values of $r$ and $\theta$. [2]

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**(b)** Hence show that $u^6$ is real and state its value. [2]

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**(c)** **(i)** On a sketch of an Argand diagram, shade the region whose points represent complex numbers $z$ satisfying the inequalities $0 \leqslant \arg(z - u) \leqslant \frac{1}{4}\pi$ and $\mathrm{Re}\, z \leqslant 2$. [4]

**(ii)** Find the greatest value of $|z|$ for points in the shaded region. Give your answer correct to 3 significant figures. [2]

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## Additional Page

If you use the following lined page to complete the answer(s) to any question(s), the question number(s) must be clearly shown.

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### 7.1 2021 | Oct/Nov | Variant 3 | 9709_w21_ms_33

Cambridge Assessment International Education

# Cambridge International AS & A Level

**MATHEMATICS** **9709/33**

Paper 3 Pure Mathematics 3 **October/November 2021**

MARK SCHEME

Maximum Mark: 75

**Published**

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

Cambridge International will not enter into discussions about these mark schemes.

Cambridge International is publishing the mark schemes for the October/November 2021 series for most Cambridge IGCSE™, Cambridge International A and AS Level components and some Cambridge O Level components.





**[Turn over**

**Generic Marking Principles**

These general marking principles must be applied by all examiners when marking candidate answers. They should be applied alongside the specific content of the mark scheme or generic level descriptors for a question. Each question paper and mark scheme will also comply with these marking principles.

GENERIC MARKING PRINCIPLE 1:
Marks must be awarded in line with:

- the specific content of the mark scheme or the generic level descriptors for the question
- the specific skills defined in the mark scheme or in the generic level descriptors for the question
- the standard of response required by a candidate as exemplified by the standardisation scripts.

GENERIC MARKING PRINCIPLE 2:
Marks awarded are always **whole marks** (not half marks, or other fractions).

GENERIC MARKING PRINCIPLE 3:
Marks must be awarded **positively**:

- marks are awarded for correct/valid answers, as defined in the mark scheme. However, credit is given for valid answers which go beyond the scope of the syllabus and mark scheme, referring to your Team Leader as appropriate
- marks are awarded when candidates clearly demonstrate what they know and can do
- marks are not deducted for errors
- marks are not deducted for omissions
- answers should only be judged on the quality of spelling, punctuation and grammar when these features are specifically assessed by the question as indicated by the mark scheme. The meaning, however, should be unambiguous.

GENERIC MARKING PRINCIPLE 4:
Rules must be applied consistently, e.g. in situations where candidates have not followed instructions or in the application of generic level descriptors.

GENERIC MARKING PRINCIPLE 5:
Marks should be awarded using the full range of marks defined in the mark scheme for the question (however; the use of the full mark range may be limited according to the quality of the candidate responses seen).

GENERIC MARKING PRINCIPLE 6:
Marks awarded are based solely on the requirements as defined in the mark scheme. Marks should not be awarded with grade thresholds or grade descriptors in mind.

| Mathematics Specific Marking Principles | |
|---|---|
| 1 | Unless a particular method has been specified in the question, full marks may be awarded for any correct method. However, if a calculation is required then no marks will be awarded for a scale drawing. |
| 2 | Unless specified in the question, answers may be given as fractions, decimals or in standard form. Ignore superfluous zeros, provided that the degree of accuracy is not affected. |
| 3 | Allow alternative conventions for notation if used consistently throughout the paper, e.g. commas being used as decimal points. |
| 4 | Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored (isw). |
| 5 | Where a candidate has misread a number in the question and used that value consistently throughout, provided that number does not alter the difficulty or the method required, award all marks earned and deduct just 1 mark for the misread. |
| 6 | Recovery within working is allowed, e.g. a notation error in the working where the following line of working makes the candidate's intent clear. |

**Mark Scheme Notes**

The following notes are intended to aid interpretation of mark schemes in general, but individual mark schemes may include marks awarded for specific reasons outside the scope of these notes.

**Types of mark**

**M** Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.

**A** Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).

**B** Mark for a correct result or statement independent of method marks.

**DM** or **DB** When a part of a question has two or more 'method' steps, the M marks are generally independent unless the scheme specifically says otherwise; and similarly, when there are several B marks allocated. The notation DM or DB is used to indicate that a particular M or B mark is dependent on an earlier M or B (asterisked) mark in the scheme. When two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.

**FT** Implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A or B marks are given for correct work only.

- A or B marks are given for correct work only (not for results obtained from incorrect working) unless follow through is allowed (see abbreviation FT above).
- For a numerical answer, allow the A or B mark if the answer is correct to 3 significant figures or would be correct to 3 significant figures if rounded (1 decimal place for angles in degrees).
- The total number of marks available for each question is shown at the bottom of the Marks column.
- Wrong or missing units in an answer should not result in loss of marks unless the guidance indicates otherwise.
- Square brackets [ ] around text or numbers show extra information not needed for the mark to be awarded.

**Abbreviations**

| | |
|---|---|
| AEF/OE | Any Equivalent Form (of answer is equally acceptable) / Or Equivalent |
| AG | Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid) |
| CAO | Correct Answer Only (emphasising that no 'follow through' from a previous error is allowed) |
| CWO | Correct Working Only |
| ISW | Ignore Subsequent Working |
| SOI | Seen Or Implied |
| SC | Special Case (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance) |
| WWW | Without Wrong Working |
| AWRT | Answer Which Rounds To |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 1 | Commence division and reach partial quotient of the form $2x^2 + kx$ | **M1** | |
| | Obtain quotient $2x^2 + 2x - 2$ | **A1** | |
| | Obtain remainder $-6x + 5$ | **A1** | |
| | | **3** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 2(a) | Show a recognizable sketch graph of $y = \lvert 2x - 3 \rvert$ | **B1** | |
| | | **1** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 2(b) | Find $x$-coordinate of intersection with $y = 3x + 2$ | **M1** | |
| | Obtain $x = \frac{1}{5}$ | **A1** | |
| | State final answer $x > \frac{1}{5}$ only | **A1** | |
| | **Alternative method for Question 2(b)** | | |
| | Solve the linear inequality $3 - 2x < 3x + 2$, or corresponding equation | **M1** | |
| | Obtain critical value $x = \frac{1}{5}$ | **A1** | |
| | State final answer $x > \frac{1}{5}$ only | **A1** | |
| | **Alternative method for Question 2(b)** | | |
| | Solve the quadratic inequality $(2x-3)^2 < (3x+2)^2$, or corresponding equation | **M1** | |
| | Obtain critical value $x = \frac{1}{5}$ | **A1** | |
| | State final answer $x > \frac{1}{5}$ only | **A1** | |
| | | **3** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 3 | Use laws of indices correctly and solve for $4^x$ | **M1** | |
| | Obtain correct solution in any form, e.g. $4^x = \frac{256}{15}$ | **A1** | |
| | Use a correct method for solving an equation of the form $4^x = a$, where $a > 0$ | **M1** | |
| | Obtain answer 2.047 | **A1** | |
| | | **4** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 4 | Commence integration and reach $ax\cos\frac{1}{2}x + b\int\cos\frac{1}{2}x\,\mathrm{d}x$ | ***M1** | |
| | Obtain $-2x\cos\frac{1}{2}x + 2\int\cos\frac{1}{2}x\,\mathrm{d}x$ | **A1** | OE |
| | Complete integration obtaining $-2x\cos\frac{1}{2}x + 4\sin\frac{1}{2}x$ | **A1** | OE |
| | Use limits correctly, having integrated twice | **DM1** | |
| | Obtain answer $2 + \frac{\sqrt{3}}{3}\pi$, or exact equivalent | **A1** | |
| | | **5** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 5 | Use double angle formula and obtain an equation in $\sin\theta$ | **M1** | |
| | Reduce to $6\sin^2\theta + \sin\theta - 5 = 0$, or 3-term equivalent | **A1** | |
| | Solve a 3-term quadratic in $\sin\theta$ and calculate $\theta$ | **M1** | |
| | Obtain answer, e.g. $56.4°$ | **A1** | |
| | Obtain second and third answers, e.g. $123.6°$ and $270°$ and no others in the given interval | **A1** | Ignore answers outside the interval. Treat answers in radians as a misread. |
| | | **5** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 6(a) | Use $\cos(A-B)$ formula and obtain an expression in terms of $\sin x$ and $\cos x$ | **M1** | |
| | Collect terms and reach $2\cos x + \sqrt{3}\sin x$ | **A1** | |
| | State $R = \sqrt{7}$ | **A1** | |
| | Use trig formula to find $\alpha$ | **M1** | |
| | Obtain $\alpha = 40.89°$ | **A1** | |
| | | **5** | |
| 6(b) | Use correct method to find $x$ | **M1** | |
| | Obtain answer $x = 220.9°$ | **A1** | |
| | | **2** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 7(a) | Use chain rule to differentiate LHS | ***M1** | |
| | Obtain $\frac{1}{x+y}\left(1+\frac{\mathrm{d}y}{\mathrm{d}x}\right)$ | **A1** | |
| | Equate derivative of LHS to $1 - 2\frac{\mathrm{d}y}{\mathrm{d}x}$ and solve for $\frac{\mathrm{d}y}{\mathrm{d}x}$ | **DM1** | |
| | Obtain the given answer correctly | **A1** | |
| | | **4** | |
| 7(b) | State $x + y = 1$ | **B1** | |
| | Obtain or imply $x - 2y = 0$ | **B1** | |
| | Obtain coordinates $x = \frac{2}{3}$ and $y = \frac{1}{3}$ | **B1** | |
| | | **3** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 8(a) | State $\overrightarrow{OM} = 4\mathbf{i} + 2\mathbf{j}$ | **B1** | |
| | Use a correct method to find $\overrightarrow{ON}$ | **M1** | |
| | Obtain answer $3\mathbf{j} + \mathbf{k}$ | **A1** | |
| | Use a correct method to find a line equation for $MN$ | **M1** | |
| | Obtain answer $\mathbf{r} = 3\mathbf{j} + \mathbf{k} + \lambda(4\mathbf{i} - \mathbf{j} - \mathbf{k})$, or equivalent | **A1** | |
| | | **5** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 8(b) | Taking a general point $P$ on $MN$, form an equation in $\lambda$ by *either* equating a relevant scalar product to zero *or* equating the derivative of $\overline{OP}$ to zero *or* using Pythagoras in triangle $OPM$ or $OPN$ | **M1** | |
| | Obtain $\lambda = \frac{2}{9}$ | **A1** | OE |
| | Use correct method to find $OP$ | **M1** | |
| | Obtain the given answer correctly | **A1** | |
| | **Alternative method to Question 8(b)** | | |
| | Use a scalar product to find the projection of $OM$ (or $ON$) on $MN$ | **M1** | |
| | Obtain answer $\frac{14}{\sqrt{18}}$ $\left(\text{or } \frac{4}{\sqrt{18}}\right)$ | **A1** | |
| | Use Pythagoras to obtain the perpendicular | **M1** | |
| | Obtain the given answer correctly | **A1** | |
| | | **4** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 9(a) | Use quotient or product rule | **M1** | |
| | Obtain correct derivative in any form | **A1** | |
| | Equate derivative to zero and solve for $x$ | **M1** | |
| | Obtain answer $x = 3$ | **A1** | |
| | | **4** | |
| 9(b) | State $\frac{du}{dx} = \frac{1}{2\sqrt{x}}$, or $dx = 2\sqrt{x}du$, or $2u\,du = dx$ | **B1** | |
| | Substitute and obtain integrand $\frac{2}{9-u^2}$ | **B1** | |
| | Use given formula for the integral or integrate relevant partial fractions | **M1** | |
| | Obtain integral $\frac{1}{3}\ln\left(\frac{3+u}{3-u}\right)$ | **A1** | OE |
| | Use limits $u = 0$ and $u = 2$ correctly | **M1** | |
| | Obtain the given answer correctly | **A1** | |
| | | **6** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 10(a) | State or imply equation of the form $\frac{dx}{dt} = k\frac{x}{20-x}$ | **M1** | |
| | Obtain $k = 19$ | **A1** | **AG** |
| | | **2** | |
| 10(b) | Separate variables and integrate at least one side | **M1** | |
| | Obtain terms $20 \ln x - x$ and $19t$, or equivalent | **A1 A1** | |
| | Evaluate a constant or use $t = 0$ and $x = 1$ as limits in a solution containing terms $a \ln x$ and $bt$ | **M1** | |
| | Substitute $t = 1$ and rearrange the equation in the given form | **A1** | **AG** |
| | | **5** | |
| 10(c) | Use $x_{n+1} = e^{0.9+0.05x_n}$ correctly at least once | **M1** | |
| | Obtain final answer $x = 2.83$ | **A1** | |
| | Show sufficient iterations to 4 decimal places to justify 2.83 to 2 d.p. or show there is a sign change in the interval (2.825, 2.835) | **A1** | |
| | | **3** | |
| 10(d) | Set $x = 20$ and obtain answer $t = 2.15$ | **B1** | |
| | | **1** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 11(a) | State or imply $r = 2$ | **B1** | |
| | State or imply $\theta = \frac{5}{6}\pi$ | **B1** | |
| | | **2** | |
| 11(b) | Use a correct method for finding the modulus or argument of $u^6$ | **M1** | |
| | Show correctly that $u^6$ is real and has value -64 | **A1** | |
| | | **2** | |
| 11(c)(i) | Show half lines from the point representing $-\sqrt{3} + \mathrm{i}$ | **B1** | |
| | Show correct half lines | **B1** | |
| | Show the line $x = 2$ in the first quadrant | **B1** | |
| | Shade the correct region | **B1** | |
| | | **4** | |
| 11(c)(ii) | Carry out a correct method to find the greatest value of $\lvert z \rvert$ | **M1** | |
| | Obtain answer 5.14 | **A1** | |
| | | **2** | |

# A Topical Questions Tracker

## A.1 Pure Mathematics 3

### A.1.1 Algebra

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9709_s21_qp_31 Question: 1 QP Page: 42 MS Page: 67
9709_s21_qp_32 Question: 1 QP Page: 80 MS Page: 105
9709_s21_qp_32 Question: 9 a QP Page: 90 MS Page: 114
9709_s21_qp_32 Question: 9 b QP Page: 91 MS Page: 114
9709_s21_qp_33 Question: 1 QP Page: 121 MS Page: 146
9709_s21_qp_33 Question: 4 a QP Page: 124 MS Page: 148
9709_w21_qp_31 Question: 6 QP Page: 161 MS Page: 182
9709_w21_qp_32 Question: 2 QP Page: 192 MS Page: 216
9709_w21_qp_32 Question: 4 QP Page: 194 MS Page: 218
9709_w21_qp_33 Question: 1 QP Page: 229 MS Page: 254
9709_w21_qp_33 Question: 2 a QP Page: 230 MS Page: 254
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### A.1.2 Logarithmic and exponential functions

9709_m21_qp_32 Question: 1 QP Page: 6 MS Page: 30
9709_s21_qp_32 Question: 3 a QP Page: 82 MS Page: 106
9709_s21_qp_32 Question: 3 b QP Page: 82 MS Page: 107
9709_s21_qp_33 Question: 2 QP Page: 122 MS Page: 146
9709_w21_qp_31 Question: 1 QP Page: 156 MS Page: 177
9709_w21_qp_32 Question: 1 QP Page: 191 MS Page: 215
9709_w21_qp_33 Question: 3 QP Page: 231 MS Page: 256

### A.1.3 Trigonometry

9709_m21_qp_32 Question: 3 QP Page: 8 MS Page: 31
9709_m21_qp_32 Question: 5 a QP Page: 11 MS Page: 32
9709_m21_qp_32 Question: 5 b QP Page: 12 MS Page: 32
9709_s21_qp_31 Question: 3 a QP Page: 44 MS Page: 69
9709_s21_qp_31 Question: 3 b QP Page: 44 MS Page: 69
9709_s21_qp_31 Question: 4 a QP Page: 45 MS Page: 69
9709_s21_qp_32 Question: 6 a QP Page: 85 MS Page: 110
9709_s21_qp_32 Question: 10 a QP Page: 92 MS Page: 115
9709_s21_qp_33 Question: 5 a QP Page: 125 MS Page: 148
9709_s21_qp_33 Question: 5 b QP Page: 126 MS Page: 149
9709_w21_qp_31 Question: 2 a QP Page: 157 MS Page: 178
9709_w21_qp_31 Question: 2 b QP Page: 157 MS Page: 178
9709_w21_qp_31 Question: 5 a QP Page: 160 MS Page: 181
9709_w21_qp_31 Question: 5 b QP Page: 160 MS Page: 181
9709_w21_qp_32 Question: 6 a QP Page: 196 MS Page: 220
9709_w21_qp_32 Question: 8 a QP Page: 200 MS Page: 222
9709_w21_qp_32 Question: 8 b QP Page: 201 MS Page: 222
9709_w21_qp_33 Question: 5 QP Page: 233 MS Page: 257
9709_w21_qp_33 Question: 6 a QP Page: 234 MS Page: 257
9709_w21_qp_33 Question: 6 b QP Page: 234 MS Page: 257

### A.1.4 Differentiation

9709_m21_qp_32 Question: 10 b QP Page: 22 MS Page: 39
9709_s21_qp_31 Question: 6 a QP Page: 48 MS Page: 72
9709_s21_qp_31 Question: 6 b QP Page: 49 MS Page: 73
9709_s21_qp_31 Question: 7 a QP Page: 50 MS Page: 73
9709_s21_qp_31 Question: 9 a QP Page: 54 MS Page: 75
9709_s21_qp_32 Question: 8 QP Page: 88 MS Page: 113
9709_s21_qp_33 Question: 3 a QP Page: 123 MS Page: 147
9709_s21_qp_33 Question: 3 b QP Page: 123 MS Page: 147
9709_s21_qp_33 Question: 8 a QP Page: 131 MS Page: 151
9709_w21_qp_31 Question: 3 a QP Page: 158 MS Page: 179
9709_w21_qp_31 Question: 3 b QP Page: 158 MS Page: 179
9709_w21_qp_32 Question: 9 a QP Page: 202 MS Page: 223
9709_w21_qp_32 Question: 9 b QP Page: 203 MS Page: 224
9709_w21_qp_32 Question: 11 a QP Page: 206 MS Page: 225
9709_w21_qp_32 Question: 11 b QP Page: 206 MS Page: 226
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9709_w21_qp_33 Question: 7 b QP Page: 236 MS Page: 258
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### A.1.5 Integration

9709_m21_qp_32 Question: 6 b QP Page: 14 MS Page: 33
9709_m21_qp_32 Question: 10 a QP Page: 21 MS Page: 39
9709_s21_qp_31 Question: 4 b QP Page: 45 MS Page: 70
9709_s21_qp_31 Question: 9 b QP Page: 55 MS Page: 76
9709_s21_qp_32 Question: 4 QP Page: 83 MS Page: 107
9709_s21_qp_32 Question: 6 b QP Page: 85 MS Page: 111
9709_s21_qp_33 Question: 4 b QP Page: 124 MS Page: 148
9709_s21_qp_33 Question: 8 b QP Page: 132 MS Page: 151
9709_w21_qp_31 Question: 4 QP Page: 159 MS Page: 180
9709_w21_qp_31 Question: 8 a QP Page: 164 MS Page: 184
9709_w21_qp_31 Question: 8 b QP Page: 165 MS Page: 184
9709_w21_qp_32 Question: 6 b QP Page: 197 MS Page: 220
9709_w21_qp_33 Question: 4 QP Page: 232 MS Page: 256
9709_w21_qp_33 Question: 9 b QP Page: 240 MS Page: 260

### A.1.6 Numerical solution of equations

9709_m21_qp_32 Question: 9 a QP Page: 19 MS Page: 37
9709_m21_qp_32 Question: 9 b QP Page: 19 MS Page: 37
9709_m21_qp_32 Question: 9 c QP Page: 20 MS Page: 38
9709_s21_qp_31 Question: 2 QP Page: 43 MS Page: 68
9709_s21_qp_31 Question: 7 b QP Page: 51 MS Page: 74
9709_s21_qp_31 Question: 7 c QP Page: 51 MS Page: 74
9709_s21_qp_32 Question: 10 b QP Page: 92 MS Page: 115
9709_s21_qp_32 Question: 10 c QP Page: 93 MS Page: 115
9709_s21_qp_32 Question: 10 d QP Page: 93 MS Page: 116
9709_s21_qp_33 Question: 6 a QP Page: 127 MS Page: 149
9709_s21_qp_33 Question: 6 b QP Page: 127 MS Page: 149
9709_s21_qp_33 Question: 6 c QP Page: 128 MS Page: 149
9709_w21_qp_31 Question: 8 c QP Page: 165 MS Page: 184
9709_w21_qp_32 Question: 11 c QP Page: 207 MS Page: 226
9709_w21_qp_33 Question: 10 c QP Page: 242 MS Page: 261

### A.1.7 Vectors

9709_m21_qp_32 Question: 7 a QP Page: 15 MS Page: 34
9709_m21_qp_32 Question: 7 b QP Page: 16 MS Page: 34
9709_s21_qp_31 Question: 8 a QP Page: 52 MS Page: 74
9709_s21_qp_31 Question: 8 b QP Page: 53 MS Page: 75
9709_s21_qp_32 Question: 11 a QP Page: 94 MS Page: 116
9709_s21_qp_32 Question: 11 b QP Page: 95 MS Page: 117
9709_s21_qp_33 Question: 9 a QP Page: 133 MS Page: 152
9709_s21_qp_33 Question: 9 b QP Page: 133 MS Page: 152
9709_s21_qp_33 Question: 9 c QP Page: 134 MS Page: 152
9709_w21_qp_31 Question: 9 a QP Page: 166 MS Page: 185
9709_w21_qp_31 Question: 9 b QP Page: 166 MS Page: 185
9709_w21_qp_31 Question: 9 c QP Page: 167 MS Page: 186
9709_w21_qp_32 Question: 10 a QP Page: 204 MS Page: 224
9709_w21_qp_32 Question: 10 b QP Page: 204 MS Page: 224
9709_w21_qp_32 Question: 10 c QP Page: 205 MS Page: 225
9709_w21_qp_33 Question: 8 a QP Page: 237 MS Page: 258
9709_w21_qp_33 Question: 8 b QP Page: 238 MS Page: 259

### A.1.8 Differential equations

9709_m21_qp_32 Question: 4 a QP Page: 9 MS Page: 31
9709_m21_qp_32 Question: 4 b QP Page: 10 MS Page: 32
9709_s21_qp_31 Question: 10 QP Page: 56 MS Page: 77
9709_s21_qp_32 Question: 7 QP Page: 86 MS Page: 112
9709_s21_qp_33 Question: 7 a QP Page: 129 MS Page: 150
9709_s21_qp_33 Question: 7 b QP Page: 130 MS Page: 150
9709_w21_qp_31 Question: 7 a QP Page: 162 MS Page: 183
9709_w21_qp_31 Question: 7 b QP Page: 162 MS Page: 183
9709_w21_qp_31 Question: 7 c QP Page: 163 MS Page: 183
9709_w21_qp_32 Question: 7 QP Page: 198 MS Page: 221
9709_w21_qp_33 Question: 10 a QP Page: 241 MS Page: 261
9709_w21_qp_33 Question: 10 b QP Page: 241 MS Page: 261
9709_w21_qp_33 Question: 10 d QP Page: 242 MS Page: 261

### A.1.9 Complex numbers

9709_m21_qp_32 Question: 8 a QP Page: 17 MS Page: 35
9709_m21_qp_32 Question: 8 b QP Page: 17 MS Page: 35
9709_m21_qp_32 Question: 8 c QP Page: 18 MS Page: 35
9709_m21_qp_32 Question: 8 d QP Page: 18 MS Page: 36
9709_s21_qp_31 Question: 5 a QP Page: 46 MS Page: 70
9709_s21_qp_31 Question: 5 b QP Page: 46 MS Page: 70
9709_s21_qp_31 Question: 5 c QP Page: 47 MS Page: 71
9709_s21_qp_32 Question: 2 QP Page: 81 MS Page: 106
9709_s21_qp_32 Question: 5 QP Page: 84 MS Page: 109
9709_s21_qp_33 Question: 10 a QP Page: 135 MS Page: 153
9709_s21_qp_33 Question: 10 b QP Page: 135 MS Page: 153
9709_w21_qp_31 Question: 10 a QP Page: 168 MS Page: 187
9709_w21_qp_31 Question: 10 b QP Page: 168 MS Page: 187
9709_w21_qp_31 Question: 10 c QP Page: 169 MS Page: 187
9709_w21_qp_31 Question: 10 d QP Page: 169 MS Page: 187
9709_w21_qp_32 Question: 3 a QP Page: 193 MS Page: 217
9709_w21_qp_32 Question: 3 b QP Page: 193 MS Page: 217
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9709_w21_qp_32 Question: 5 b QP Page: 195 MS Page: 219
9709_w21_qp_33 Question: 11 a QP Page: 243 MS Page: 262
9709_w21_qp_33 Question: 11 b QP Page: 243 MS Page: 262
9709_w21_qp_33 Question: 11 c QP Page: 244 MS Page: 262